Stationary and quasi-stationary solutions of the Burgers type equations

Mikhail Averianov, Marina Basova, Vera Khokhlova

Department of Acoustics, Faculty of Physics, M. V. Lomonosov Moscow State University, Leninskie gory, Moscow 119992, Russia, Email: vera@acs366.phys.msu.ru

Introduction

Nonlinear acoustical models based on the evolution equations of Burgers and Khokhlov-Zabolotskaya-type have been widely used in medical applications, geophysics, and nonlinear diagnostics [1 - 4]. Various frequency dependence of absorption, typical for media with complex internal structure, has a strong effect on nonlinear wave propagation [3, 4]. Different absorption terms are also used in numerical simulations to provide the stability of algorithms of modeling shock waves. This additional absorption may result in nonphysical effects. It is important, therefore, to study basic nonlinear wave phenomena, related to the presence of frequency dependent absorption. The solutions in the form of stationary shock wave or quasi-stationary periodic wave are studied in this work for the Burgers equation generalized for different power law of absorption.

Theory

Governing equations

It is well known, that the effects of acoustic nonlinearity lead to the steepening of the wave front. When a resulting shock front becomes sufficiently narrow, the smoothing dissipation effect becomes stronger [1]. The combined effects of nonlinearity and absorption result in formation of the stationary shock front. The structure of the shock depends on the law of frequency dependent absorption and dispersion.

Consider propagation of nonlinear wave in the medium governed by the Burgers equation with an absorption term in the form of the second, forth, or sixth derivative. In dimensionless coordinates the equation can be written as

$$\frac{\partial V}{\partial z} - V\frac{\partial V}{\partial \theta} = (-1)^{\frac{\eta-2}{2}}\,\Gamma\,\frac{\partial^{\eta} V}{\partial \theta^{\eta}} \quad . \tag{1}$$

$$V = p/p_0\,,\quad \theta = \omega_0 \tau\,,\quad \tau = t - x/c_0\,,\quad \eta = 2, 4, 6$$

$$z = \frac{x}{x_s} = \frac{\varepsilon p_0 \omega_0}{\rho_0 c_0^{\,3}}\,x\,,\quad \Gamma = \frac{\beta \omega_0^{\eta-1} c_0^3 \rho_0}{\varepsilon p_0}\,. \tag{2}$$

Here V is the sound pressure, normalized by the initial wave amplitude p_0, θ is the dimensionless time in retarded coordinate system, x_s is the shock formation distance for harmonic wave with frequency ω_0, Γ is the dimensionless parameter of absorption, similar to the inverse acoustic Reynolds number in the Burgers equation [1].

Self-similarity of the solutions

The solutions of the Eq. (1) have self-similar character, which allows us to obtain for each parameter η a set of solutions of this equation based on only one known solution. It is easy to show, that, if the function $V_0 \cdot V(\theta_0, z_0)$ is a solution of Eq. (1) in the coordinates θ_0 and z_0 for given absorption parameter Γ_0 and amplitude V_0, then the function $V_1 \cdot V(\theta_1, z_1,)$, written in new variables

$$z_1 = z_0 \frac{V_1}{V_0} \sqrt[\eta-1]{\frac{\Gamma_0}{\Gamma_1}\frac{V_1}{V_0}} \qquad \theta_1 = \theta_0 \sqrt[\eta-1]{\frac{\Gamma_0}{\Gamma_1}\frac{V_1}{V_0}}\,, \tag{3}$$

is a solution too. Here Γ_1 and V_1 are arbitrary new absorption and wave amplitude values. This self-similar property of the solutions can be applied to investigate the effect of absorption on nonlinear propagation of acoustic waves.

Energy absorption

Absorption of the acoustic wave energy $E = \int V^2 d\theta$ at the finite width shock front of the stationary nonlinear wave does not depend on the value of absorption parameter Γ for $\eta = 2$ and 4, it is proportional to the shock amplitude in the third power, and is equal to the energy absorption at the infinitely thin shock [1, 3]. It can be shown, that the same relation is valid for energy losses on the shocks in case of absorption, proportional to the arbitrary even derivative, in particular, when $\eta = 6$:

$$\frac{dE}{dz} = \int_{-V_0/2}^{V_0/2} (V^2 - V_0^2/4)\,dV = -\frac{1}{6}V_0^3\,, \tag{4}$$

where V_0 is the amplitude of the shock. This law can be also applied to the propagation of quasi-stationary nonlinear waves at the stage of well-developed shocks with the width much smaller that the characteristic wave period.

Results

The simulations of the Burgers-type Eq. (1) have been performed in this work for $\eta = 2, 4, 6$ and for initially step-like waveform, harmonic wave, or gaussian signal. Stationary solutions in the form of the shock wave and quasi-stationary solutions in the form of the periodic sawtooth-like waves or triangular pulses are obtained and compared for various absorption laws. The dynamics of the nonlinear waveform and shock front structure is investigated depending on the travel distance and absorption parameters.

Stationary solutions

In the case of initial step-like signal and viscous fluid ($\eta = 2$, Burgers equation) the stationary solution has the form of the smooth shock front structure [1]. For higher order of derivative of the absorption term, $\eta = 4$, the structure of the shock front in the stationary solution is different: oscillatory behavior is observed around the front [3]. The oscillations are even more pronounced in the case of absorption represented by the sixth order derivative, $\eta = 6$ (Figure 1).

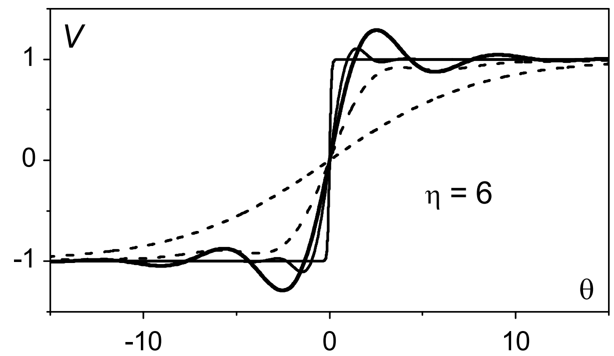

Figure 1: Evolution of an initial smooth (dashed line) and steep (solid line) signal to the stationary waveform (thick solid line) calculated for η=6 and Γ = 1.0.

The steepness of the shock front, which is inverse proportional to the front width, is one of the most important characteristics of the shock wave. Self-similar properties of the solutions of Eq. 1 and stationary waveforms, obtained numerically for $\Gamma = 1.0$ lead to the following dependence of the steepness of the shock front $\Delta^{-1} = dV/d\theta(\theta = 0)$ in the stationary solution on the absorption parameter Γ (Figure 2): $\Delta^{-1}(\Gamma) = \Gamma^{-1/\eta-1} \cdot \Delta^{-1}(\Gamma = 1)$. Here the shock amplitude is considered to be constant, $V_0 = 2$. It is seen, that for smaller absorption $\Gamma < 0.5$, the shock front is steeper for lower order of derivative of the absorption term, the dependence on η is the opposite for higher values of absorption, $\Gamma > 1$, and shock steepness is about equal for all η within the range $\Gamma =$ 0.5 – 0.8 ($\Delta_\eta^{-1} \sim 1.0$), where the three curves intersect.

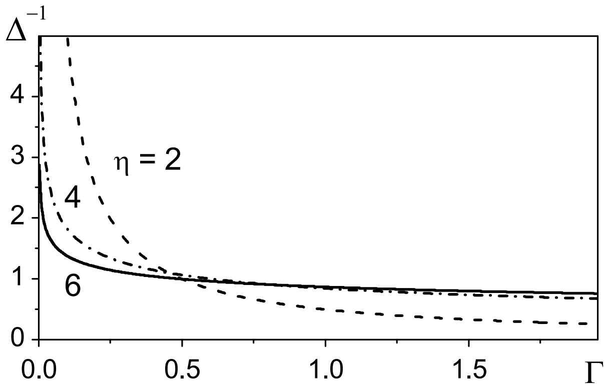

Figure 2: Dependence of the shock front steepness in the stationary solution of Eq. 1 on the absorption parameter Γ for various power law of absorption $\eta = 2, 4, 6$.

Quasi-stationary solutions

Simulations have been performed for propagation of an initially harmonic wave and gaussian pulse in the media with various absorption law η and various values of parameter Γ. It is shown that for small values of absorption Γ, the evolution of the waveform at the distances of well-developed shocks (quasi-stationary waves) is in a good agreement with the solutions of the simple wave equation modified for the shock front structure by the corresponding stationary solution. The oscillatory behavior is observed around the shock front of quasi-stationary solutions for η = 4, 6. The steepness of the shock front changes with the propagation distance due to the dependence on the wave amplitude. Shown in Figure 3 are the propagation curves for the maximum steepness of the waveform, calculated from the numerical solutions of Eq. 1 for initially harmonic wave and different absorption laws. Absorption parameters Γ_η in simulations were chosen according to the stationary solutions so that the steepness of the front is equal (Δ_0^{-1}=5.0) for all values of η, when the amplitude of the sawtooth wave decreases twice ($A_0 = 1$) comparing with the maximum shock amplitude: $\Gamma_\eta = \Gamma(A_0/A_s)^\eta(\Delta_\eta^{-1}/\Delta_0^{-1})^{\eta-1}$, $\Gamma_2 = 2.5\cdot10^{-2}$, $\Gamma_4 = 3\cdot10^{-4}$, and $\Gamma_6 = 2.45\cdot10^{-6}$. Here Δ_η^{-1} are known values of stationary solutions for Γ=1.0 and $A_s = 2.0$. The results of simulations for initially harmonic wave are shown in the Figure 3. It is seen that for all values of η the maximum of the steepness is achieved at the distance $z \sim \pi/2$, which corresponds to the maximum value of the shock amplitude. Shock front smoothening is stronger pronounced for lower power law of absorption.

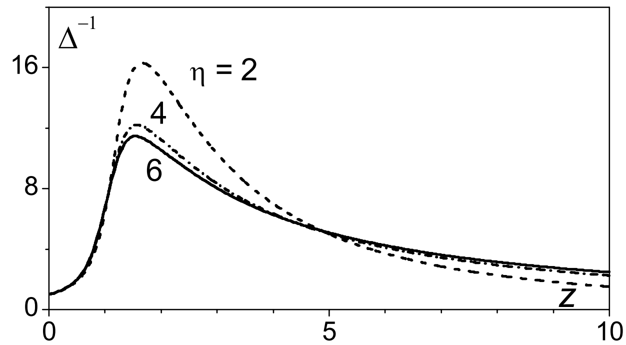

Figure 3: Dependence of the shock front steepness on the propagation distance for initially harmonic wave.

Acknowledgments

This work was partly supported by the ONRIFO, CRDF, and RFBR grants.

References

[1] O.V. Rudenko and S.I. Soluyan. Theoretical Foundations of Nonlinear Acoustics. Plenum, New York, 1977

[2] L.A. Chernov. Waves in randomly inhomogeneous media. Moscow, Nauka, 1975

[3] O.V. Rudenko, V.A. Robsman. Nonlinear wave equation in scattering medium. Doklady Akademii Nauk **384**(6) (2002), 1-5

[4] E.A. Filonenko, V.A. Khokhlova. Effect of acoustic nonlinearity on heating of biological tissue induced by high intensity focused ultrasound. Acoust. Phys. **47**(4) (2001), 468-475