Period doubling on cylindrical reed instruments

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Introduction

Period doubling phenomena are expected to be possible on self-sustained instruments, at least since the famous paper by Mc Intyre et al [1]. Some theoretical papers have been written concerning cylindrical reed (i.e. clarinet-like) instruments : Kergomard [2],[3] has shown that for the simple model based upon the Bernoulli equation, ignoring losses in the resonator, period doubling, tripling can occur. On the experimental point of view, if such a behaviour has been observed on a bassoon, it is difficult to obtain for cylindrical reed instruments. Nevertheless, using a crumhorn with a soft plastic reed, Gibiat and Castellengo [4] got period doubling.

The first aim of this paper is to extend the theoretical analysis when resonator losses are taken into account : analytical calculation when losses are independent of the frequency are possible, the corresponding model being called "Raman model", by analogy with the work of Raman on bowed string. The second aim is to confirm the previous result by using a real time synthesis method [5].

Simple model

When the reed is assumed to act as a simple spring without inertia, which is reasonable at least for the steady-state regime, it is well known [1] that the functioning of a reed instrument can be modeled using only two equations, written e.g.

$$p(t) = [\widehat{z} * u]\,(t) \quad ; \quad u(t) = F\,[p(t)]\,. \qquad (1)$$

where $p(t)$ and $u(t)$ are the acoustic pressure and volume velocity, respectively, at the input of the instrument. $\widehat{z}(t)$ is the inverse Fourier Transform of the input impedance of the resonator. F is a nonlinear function characterizing the excitation, which can be written, using dimensionless quantities [2], as follows :

$$u = \zeta(1-\gamma+p)\sqrt{\gamma-p}. \qquad (2)$$

if $p > \gamma - 1$ and $u = 0$ if $p < \gamma - 1$. γ is the static pressure in the mouth, which is the source of energy (divided by the minimum pressure, p_M, for which the reed channel is closed in static regime), and ζ is a parameter of the embouchure, equal to $\zeta = \sigma(2\gamma_m p_a/p_M)^{\frac{1}{2}}$: σ is the area ratio of the reed opening (at rest) and the tube, γ_m the ratio of the specific heats, and p_a the atmospheric pressure (ζ is also related to the maximum volume velocity which can enter in the tube). Despite its simplicity, the model is rather realistic (see e.g. [6]).

For a cylindrical instrument of length ℓ, if a zero impedance is assumed at the end (this approximation is not discussed here), the equation for the resonator can be written as follows :

$$p_n - u_n = -(p_{n-1} + u_{n-1}) \qquad (3)$$

if at each time $t = n2\ell/c$, we use the notation $p_n = p(n2\ell/c)$ and $u_n = u(n2\ell/c)$. It has been shown [1] that this iteration equation, combined with the nonlinear function F leads to an iterated map scheme.

Raman model

If losses are independent of frequency, the wavenumber $k = \omega/c$ is replaced by $k = \omega/c - j\alpha/\ell$. α is independent of frequency, and is related to the Raman parameter λ by $e^{-2\alpha} = \lambda$. It can be shown that all results obtained for p, u, F when $\alpha = 0$ remain valid when $\alpha \neq 0$ by replacing them by q, w, G, respectively, where :

$$p_n = q_n \cosh(\alpha) + w_n \sinh(\alpha)\ ; \qquad (4)$$

$$u_n = q_n \sinh(\alpha) + w_n \cosh(\alpha); \qquad (5)$$

$$w_n = G(q_n)\,;\ q_n - w_n = -(q_{n-1} + w_{n-1}). \qquad (6)$$

Therefore, if function G is known, the iterated map can be generalized using equations (6). Moreover the graphical method used by Maganza et al [7] can also be used. As an example, the static regime is given by $q_n = 0$, and its stability by $dG/dq(q=0) < 0$. The analytical calculation of G is difficult in general, but its derivative is obtained from equation (4) and (5):

$$dG/dq = (dF/dp - \tanh\alpha)\,/\,(1 - dF/dp\tanh\alpha)\,. \qquad (7)$$

Stability of the periodic regime

Thanks to the simplicity of the nonlinear function, all calculations concerning the periodic regime (with two points, corresponding to a square signal) are possible analytically, including the stability. It can be checked that the low limit of stability, when γ varies, corresponds to the limit of instability of the static regime (direct bifurcation). In the present paper, the upper limit $\gamma_{doubling}$ is discussed : above this limit, a period doubling can be obtained, as shown when losses are ignored, and also other regimes with larger periods, or even chaos. We call this range of parameter γ with subharmonics spectra period-doubling range, even if the period can be larger. It is actually narrow, because the beating-reed threshold is not far (see [8]). If the reed is beating (one of the corresponding values of the volume velocity vanishes), it can be shown that in the common range of parameters, the two-points regime is always stable just above the beating threshold, which is found to be :

$$\gamma_{beating} = \frac{1}{2}\left[1 + T^2\left[1 + (1 - T^2)/\zeta^2\right]\right]. \qquad (8)$$

where $T = \tanh\alpha$. [For stronger values of the mouth pressure, two regimes can be found, with an inverse bifurcation corresponding to the saturation mechanism [8]]. If the reed is not beating, it can be shown that the period doubling limit $\gamma_{doubling}$ increases significantly when losses increase, depending on the value of the parameter $\zeta^{-1}\tanh\alpha$. This result confirms numerical (ab initio) calculation given in ref [6]. An important result is that this limit $\gamma_{doubling}$ can reach the beating one, $\gamma_{beating}$ for a rather small value of the loss parameter, found to be :

$$\tanh\alpha = \frac{1}{2}\zeta^3\left[1 - \zeta^2 + O(\zeta^4)\right]. \qquad (9)$$

For higher losses, the period-doubling range disappears. A consequence is the difficulty to get period doubling with clarinet-like instruments : it was already known that without losses the phenomenon occurs only for rather high values of ζ (the period-doubling range tends to zero when ζ tends to zero). When losses are taken into account, the phenomenon can disappear even for non small values of ζ. If it is assumed that, for frequency-dependent losses, the above calculations remain roughly valid if α is chosen to be the value for the fundamental frequency, the result can be compared with typical practical situation : if $\zeta = 0.35$, the limit value of the losses is found to be $\alpha = 0.019$. For the fundamental frequency, a typical value of α is 0.025.

Real-time synthesis

Using the method of ref. [5], we have checked the previous result. With this method, losses are "exact" only for the two first impedance peaks, and approximate for the upper peaks, but it is possible to take into account one mode of vibration of the reed. It is confirmed that above a threshold of losses the period doubling range disappears, and the values are very close to the theoretical ones, with a small influence of the reed resonance. Fig. 1 shows an example of spectrogram obtained for a kind of crescendo. The value of ζ being rather high, period doubling appears. Note that after the period doubling, the reed beats.

Other cases have been studied : as an example, fixing the value of γ, a complete route to the chaos is obtained by successive period doublings when ζ increases (see Fig. 2). This study is unfortunately not easily feasible experimentally, because the reed and mouthpiece parameters cannot be changed continuously, contrary to the mouth pressure. The value of the reed resonance frequency does not affect the previous result.

We can conclude with a question : when choosing the dimensions of the instruments and the characterics of the reed and mouthpiece, are the makers looking for "normal" sounds, avoiding both squeak and double-period sounds ? If yes, the choice of the different parameters

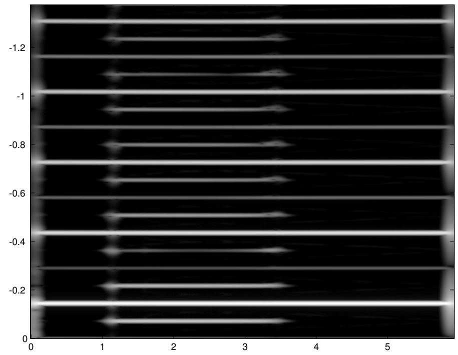

Figure 1: Spectrogram of a "crescendo" obtained by real-time synthesis for high ζ : the period doubling range appears clearly, but this result corresponds to weak losses.($\zeta = 0.6, \gamma$ varies from 0.45 to 0.55, $\alpha = 0.029$). The x-axis is time, the y-axis is frequency (in kHz).

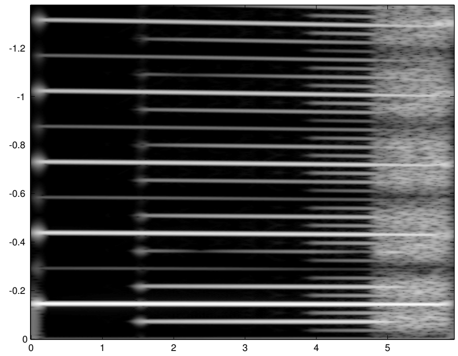

Figure 2: Variation of the parameter ζ from 0.3 to 1.1 : a route to the chaos appears ($\gamma = 0.49$, $\alpha = 0.029$). The x-axis is time, the y-axis is frequency (in kHz)

is probably rather narrow. Obviously a detailed study remains to do, including precise experiments, in order to answer deeply to the question....

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