Numerical BEM/FEM formulation applied to plane wave diffraction by elastic inclusion

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Abstract:
We study the ultrasonic wave transmission and diffraction in homogeneous and isotropic media. In this aim, we use a coupled numerical formulation by boundary and finite element methods (BEM/FEM).

The domain concerning limited inclusion is treated by finite elements, the unlimited domain is handled by boundary elements method.

The coupled formulation is validated using analytical solution. Stoneley interface mode and plane wave diffraction by cylinder are presented in this work.

Introduction

The modelling of the NDT experiments by sound waves has been the object of several works. The use of numerical methods asks for heavier means but allows raising a big number of restrictive hypotheses for analytical methods.

We present here a numerical coupling method BEM/FEM applied to elastodynamic problems.

The finite element method allow modelling diverse anisotropic elastic inclusions. The boundary element method allow implicit using of radiation condition at the infinity when unlimited homogeneous elastic domain is considered.

The coupled methods differ generally, one of the others, by the choice of the integral formulation in use, or by the consideration of the conditions of continuity...

Costabel et all [1] presented a general mathematical concept of the use of a coupled formulation BEM / FEM for the acoustoelastic harmonic problem. They use a variational formulation in displacement form FEM, and a variational direct integral representation BEM. Writing continuity conditions for displacement and stress vectors on the coupling interface in integral form allows obtaining a non-symmetric global formulation.

Polizotto [2] provides many forms of variational approach in FEM domain based on Hu-Washizu or Hellinger-Reissner instead of the variational displacement form. In BEM domain, he uses a variational approach based on indirect integral representations. All forms are used in static case, their extension in the elastodynamic does not raise particular problems.

Han-Hou [3] presents some coupled BEM/FEM formulation applied to static problems. He uses a variational in displacement form of FEM domain, and variational BEM formulation based on direct integral representation. The continuity of the normal stress to the interface between both domains, written in two constituents among which the one obtained using the complete representation, allows him to obtain a symmetric global formulation. The extension of this approach in the elastodynamic is immediate.

We resolve the interface coupled problem in elastic wave propagation, where the BEM is based on indirect approach. Classical variational approach FEM in displacement form is used. Analytical developments validates the numerical model.

Motion equations

In 2D and linear elastodynamic, we consider a body \( \Omega \) as shown in Figure 1. \( \Omega = \Omega_1 \cup \Omega_2 \), with regular boundary \( \Sigma = \Sigma_1 \cup \Sigma_2 \).

![Figure 1. Geometry of the interface problem.](image)

The governing equation for displacement vector is expressed as:

\[
div(\sigma_i) + \rho \omega^2 u_i = 0, \quad i = 1,2, \tag{1}
\]

where \( \sigma_i \) is the stress tensor in \( \Omega \):

\[
\sigma_i = C_{ij} e_j, \quad i = 1,2. \tag{2}
\]

An incident field is considered in \( \Omega_1 \), the total displacement is decomposed in \( u_{\text{inc}} \) and \( u_i \):

\[
u_i = u_i + u_{\text{inc}}. \tag{3}
\]

\( u_i \) satisfies the radiation condition at infinity. \( \Omega_1 \) and \( \Omega_2 \) are coupled by the boundary \( \Sigma_i \), and continuity is applied to total components of displacement and stress:

\[
\begin{align*}
\mathbf{u}_i(\Sigma) &= \mathbf{u}_2(\Sigma) \quad (a) \\
\mathbf{t}_i(\Sigma) &= \mathbf{t}_2(\Sigma) \quad (b) \\
M \in \Sigma_i.
\end{align*}
\]

Formulations

The coupling process used in this work is the following:

In the variational principal applied to \( \Omega_2 \) we use continuity condition (4) b thus \( t_i \) is replaced with the integral representation. The continuity condition for displacements and the boundary condition on \( \Sigma_e \) are applied to displacements in integral form for \( u_i \).

Finally, we obtain three integral equations with three unknown variables \( u_2, \psi \) on \( \Sigma_i \) and \( \sigma \) on \( \Sigma_e \).

The coupling boundary \( \Sigma_i \) is divided into linear elements (L2). A piecewise linear variation along each boundary element is assumed for unknowns \( \psi \) and \( \sigma \). The displacement \( u_i \) in \( \Omega_2 \) is interpolated with linear shape functions and (T3) element is used.

We obtain a non symmetric assembled matrix given by:

\[
\begin{bmatrix}
\mathbf{[\Sigma]} & \mathbf{[T]} \\
\mathbf{[T]} & \mathbf{[\Sigma]} & \mathbf{[G]}
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_{i;\text{inc}} \\
\mathbf{u}_i \\
\mathbf{\sigma}_{i;\text{inc}}
\end{bmatrix} =
\begin{bmatrix}
-T_{u_2} - U_{\text{inc}/z} \\
U_{\text{inc}/z} - T_{u_2} \\
U_{\text{inc}/z} - U_{\text{inc}/z}
\end{bmatrix}.
\]

\[
\begin{bmatrix}
\mathbf{[G]} & \mathbf{[T]} & \mathbf{[\Sigma]}
\end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_i \\
\mathbf{\sigma}_i
\end{bmatrix} =
\begin{bmatrix}
\mathbf{u}_{\text{inc}} \\
\mathbf{\sigma}_{\text{inc}}
\end{bmatrix}.
\]

\( \mathbf{[T]} \) is the identity matrix.
Validation

Plane wave diffraction by inclusion:

The numerical formulation is validated using analytical solution of plane wave diffraction by cylindrical homogenous inclusion [4].

We present a superposition in polar representation obtained by analytical solution and coupled numerical formulation for radial displacement modulus.

The internal domain is a circular homogenous or square homogenous inclusion. In the Figure 2a more important back scattering is shown in the case of plane interface. In Figure 2b circular inclusion is considered, so we notice that diffracted lobes are generated when inclusion stiffness is greater.

Stoneley modal wave problem in interface:

Consider two elastic half spaces in perfect contact. The usual equation of motion is resolved to find modal solution for wave propagating in the $x$ and $y$ directions that vanishes when $y$ increases [5]. For the real roots of the dispersion equation, we find (numerically) the wave number solution for a given frequency $f$. The modal solution is then injected as boundary condition on coupled domains. We compare in Figure 4, numeric and analytic displacements given in the two domains. This kind of mode is shown in the case of plane wave diffraction, independently of the chosen frequency, and of the inclusion geometry, we noticed the generation of this type of modal wave on the interface.

Conclusion

We present in this work a numerical coupled formulation by FEM and BEM applied to interface problem and wave transmission. This formulation is validated for plane wave diffraction using analytical solution. We discussed effects of stiffness, geometry and incidence angle on wave diffraction. We show interface wave generated in this case, so we have given for the case of Stoneley mode a numerical validation by the coupled method. The coupling method gives very accurate results for low meshing criterion.

References


