Two-channel resonant formalism applied to the scattering of a fluid cylinder in an elastic medium

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Introduction

The difficulties encountered in studies concerning the acoustic scattering from cylindrical or spherical fluid inclusions in a solid medium [1] are that two competing types of phenomena are closely intricated. On the one hand, the interaction of an acoustic field and the inclusion induces mode conversions between compressional and shear waves at the boundary between the inclusion and the solid medium. On the other hand, the resonances of the fluid filling the inclusion are mixed with the diffracted contributions from the wall of the cavity in the two outgoing scattering channels. As it will be seen in the following, the Multichannel Resonant Scattering Theory (MRST) formalism [2] is the accurate tool designed to separate each type of the previous contributions.

Two-channel Scattering Matrix

The two-channel scattering matrix $\mathbb{S}$ build up in the case of the scattering of a monochromatic plane wave normally incident on a fluid cylindrical cavity (radius: $r = a$; density: ρ_f ; wave velocity: c_f) in an elastic homogeneous isotropic medium (density: ρ_S ; longitudinal wave velocity: c_L ; transversal wave velocity: c_T) is

$$\mathbb{S} = \begin{bmatrix} S^{LL} & S^{LT} \\ S^{TL} & S^{TT} \end{bmatrix}. \tag{1}$$

For a given mode n , the first line components of $\mathbb{S}$, are the amplitude coefficients of the scattered scalar ϕ and vector ψ potentials respectively written as Rayleigh series

$$\begin{cases} \phi = -\left(\sum_{n=-\infty}^{\infty} j^{n-1} (S^{LL} - 1) H_n^{(1)}(K_L r) e^{jn\theta} \right) / (2\rho_S \omega^2) \\ \psi = -\left(\sum_{n=-\infty}^{\infty} j^{n-1} S^{LT} H_n^{(1)}(K_T r) e^{jn\theta} \right) / (2\rho_S \omega^2) \end{cases} \tag{2}$$

when the cavity is insonified by a longitudinal (or L) wave defined as follows via the incident scalar potential

$$\phi_{inc} = -\left(\sum_{n=-\infty}^{\infty} j^n J_n (K_L r) e^{jn\theta} \right) / (\rho_S \omega^2) . \tag{3}$$

In relations (2) and (3), ω is the pulsation of the incident wave, $K_L = \omega / c_L$ and $K_T = \omega / c_T$ are the longitudinal and transversal wave numbers in the solid respectively. S^{LL} and S^{LT} are calculated by solving the boundary equations

$$\sigma_{rr})_{solid} = -p \ , \ \ u_r)_{solid} = u_r)_{fluid} \ , \ \sigma_{r\theta})_{solid} = 0 \ , \tag{4}$$

where σ_{rr} and $\sigma_{r\theta}$ are the normal and tangential stresses, u_r is the radial displacement, and p the pressure in the fluid. The second line components of $\mathbb{S}$ are the amplitude coefficients of the scattered scalar and vector potentials in the case an incident transversal (or T) wave

$$\psi_{inc} = -\left(\sum_{n=-\infty}^{\infty} j^{n-1} J_n (K_T r) e^{jn\theta} \right) / (\rho_S \omega^2) . \tag{5}$$

S^{TL} and S^{LT} are also calculated using the boundary conditions (4). The resulting $\mathbb{S}$ matrix is symmetric and fulfils the energy conservation identity $\mathbb{S}\mathbb{S}^\dagger = \mathbb{S}^\dagger \mathbb{S} = \mathbb{I}$ (the sign † designates the adjoint operator and $\mathbb{I}$ is the identity matrix). Its diagonal components verify $|S^{LL}| = |S^{TT}|$. Its determinant is noted $S = \mathrm{e}^{-2j\delta}$.

Resonant Scattering Matrix Analysis

Background Removal

In order to isolate the n^{th} mode resonances of the fluid cylinder while preserving the unitarity, the resonant scattering matrix $\mathbb{S}^{(*)}$ is obtained by forming the product

$$\mathbb{S}^{(*)} = \mathbb{S}_0^\dagger \mathbb{S} \ , \tag{6}$$

where $\mathbb{S}_0$ is the unitary scattering matrix built up in the case of the "soft" cavity scattering. $\mathbb{S}_0$ is easily derived from $\mathbb{S}$ by setting $\rho_f = 0$, and we note its determinant $S_0 = \mathrm{e}^{-2j\delta_0}$. The moduli of $\mathbb{S}^{(*)}$ diagonal components $S^{LL(*)}$ and $S^{TT(*)}$ are equal, as the moduli of its non diagonal components $S^{LT(*)}$ and $S^{LT(*)}$ also do.

Resonant Scattering Matrix Eigenvalues

The two eigenvalues of $\mathbb{S}^{(*)}$ are found to be exactly equal to

$$S_\pm^{(*)} = \begin{cases} 1 \\ S_0^* S \end{cases} . \tag{7}$$

It means that only the (-) eigenchannel is involved in the resonant scattering process, while the (+) eigenchannel is a closed channel. All the fluid cylinder resonant energy is then contained in $S_-^{(*)} = \mathrm{e}^{-2j\delta^{(*)}}$, which eigenphase is equal to $\delta^{(*)} = \delta - \delta_0$. Then, the diagonalized form $\mathbb{S}_{diag}^{(*)}$ of $\mathbb{S}^{(*)}$ is commonly written $\mathbb{S}^{(*)}_{diag} = \mathbb{I} + 2j\mathbb{T}^{(*)}_{diag}$ using the two-channel diagonal transition matrix

$$\mathbb{T}^{(*)}_{diag} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} T^{(*)} . \tag{8}$$

As a function of the normalized frequency variable $x_L = K_L a$, the transition amplitude $T^{(*)} = (S^{(*)}_{-} - 1)/2j$ exhibits sharp resonance peaks , as shown in Figure 1 for a water-filled cylindrical cavity in aluminium.

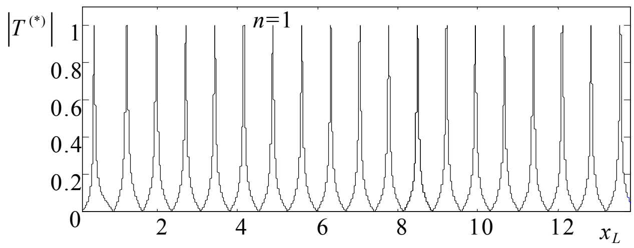

Figure 1: Modulus of $T^{(*)}$ plotted versus the normalized frequency x_L for the mode n=1, in the case of a water-filled cylindrical cavity in aluminium.

Indeed, $T^{(*)}$ can be easily approximated by the Breit-Wigner resonant form

$$T^{(*)} \approx -(\Gamma/2)/(x_L - \underset{\_}{x}^{*}_{L} + j\Gamma/2) \tag{9}$$

near the real part of a pole $\underline{x}^{*}_{L} = x^{*}_{L} - j\Gamma/2$ of $\mathbb{S}$ related to the fluid cylinder eigenmodes (see Figure 2).

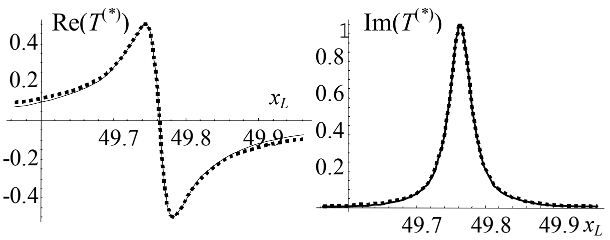

Figure 2: Comparison between $T^{(*)}$ – solid line – and the Breit-Wigner approximation formula (9) – dotted line - near the pole $\underline{x}^{*}_{L} = 49.764 - i\ 0.0191$ for the mode n=1.

Density Matrix

The change of basis from the resonant scattering $\mathbb{S}^{(*)}$ matrix to its eigenmatrix $\mathbb{S}^{(*)}_{diag}$ involves the rotation matrix $\mathbb{R}_0$

$$\mathbb{R}_0 = \begin{bmatrix} \cos\alpha\, e^{j\delta_1} & -\sin\alpha\, e^{-j\delta_2} \\ \sin\alpha\, e^{j\delta_2} & \cos\alpha\, e^{-j\delta_1} \end{bmatrix} \tag{10}$$

such that $\mathbb{S}^{(*)} = \mathbb{R}_0 \mathbb{S}^{(*)}_{diag} \mathbb{R}^{\dagger}_0$. It is worth noting that the mixing angles $\alpha \geq 0$, δ_1 and δ_2 exclusively depend on coefficients involved in the tangential stress of the boundary equation $:\sigma_{r\theta}\big)_{solid} = 0$ and do not include any resonant contributions of the fluid. In order to quantify the energy distribution between the resonant L and T channels in $\mathbb{S}^{(*)}$, it is more convenient to use the density matrix $\rho = \mathbb{R}_0 (\mathbb{T}^{(*)}_{diag} / T^{(*)}) \mathbb{R}^{\dagger}_0$. In quantum physics, the density matrix related to resonance phenomena is commonly expressed in terms of partial resonance widths [2] $\Gamma_1 = \Gamma\cos^2\alpha$ and $\Gamma_2 = \Gamma\sin^2\alpha$, leading to

$$\rho = \begin{bmatrix} \dfrac{\Gamma_2}{\Gamma} & -\dfrac{\Gamma_1^{1/2}\Gamma_2^{1/2}}{\Gamma} e^{-j\gamma} \\ -\dfrac{\Gamma_1^{1/2}\Gamma_2^{1/2}}{\Gamma} e^{j\gamma} & \dfrac{\Gamma_1}{\Gamma} \end{bmatrix} \tag{11}$$

with $\gamma = \delta_2 - \delta_1$. The relation between the resonant transition matrix $\mathbb{T}^{(*)}$ in the original basis and the resonant amplitude $T^{(*)}$ simply becomes

$$\mathbb{T}^{(*)} = (\mathbb{S}^{(*)} - \mathbb{I})/2j = \rho T^{(*)} . \tag{12}$$

In Figure 3, the moduli of the diagonal components $T^{LL(*)}$ and $T^{LL(*)}$ of $\mathbb{T}^{(*)}$ are plotted versus x_L ,each one being compared to the plot of the related diagonal component of ρ .

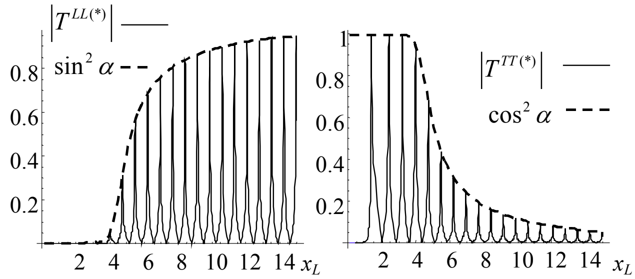

Figure 3: $\mathbb{T}^{(*)}$ diagonal components moduli compared to the plots of the diagonal components of the density matrix ρ versus x_L , for the mode n=10.

Finally, $\mathbb{S}$ is the superimposition of two terms: the first term is a translation only depending on the soft background, and the second term represents the projection of the resonances over each channel by means of the matrix $\mathbb{S}_0\rho$, i.e.

$$\mathbb{S} = \mathbb{S}_0 + \mathbb{S}_0 \rho T^{(*)} . \tag{13}$$

Conclusion

It is important to recall that if the M.R.S.T. provides a simple and clear understanding of the several contributions occuring in such a scattering problem, the main difficulty lies in the preliminary choice of the background to be factorized. The previous work can be fully transposed to the case of fluid-filled spherical cavity.

References

[1] S.G. Solomon, H. Überall and K.B. Yoo, Acustica 55, 147–159 (1984).

[2] E. B. Danila, J. M. Conoir, P. Pareige and J. L. Izbicki , Wave Motion 28, 297-318 (1998).