Mixed Spectral Finite Elements for Computational Acoustics

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Introduction

In computational acoustics one has to deal with large scale problems such as noise barriers or the acoustic far field radiation of transformers. In contrast to this, the linear finite element method (*h-FEM*) requires very fine simulation grids.

One idea to overcome this problem is the application of higher order finite element methods such as the hierarchical p finite element method (*p-FEM*)[5]. Here we will investigate an alternative approach, the so called spectral finite element method (*spectral-FEM*) as introduced in [1].

The efficient application of this method to the governing equations of acoustics has been introduced in [3]. As this is a much more general approach, it can be easily extended. We will present the basic relations and some numerical results showing the very good approximation features of this method.

Spectral Element Method

The use of higher order methods is motivated by the smoothness of solutions to the wave equation. In fact, the discretization error is reduced exponentially while increasing the order of approximation as shown in [1]. Spectral finite element methods use in general Lagrange basis functions of arbitrary order N

$$\mathcal{P}_i^N(\xi) = \frac{\prod_{i=0,i\neq j}^{N} \xi - \xi_i}{\prod_{i=0,i\neq j}^{N} \xi_i - \xi_j} \,. \tag{1}$$

The key point in the spectral element method is to choose the supporting points not arbitrarily but at the zeros of Legendre or Chebychev polynomials. Here Legendre polynomials are chosen, because the supporting points coincide with the integration points of the Gauss-Lobatto quadrature rule which leads to very efficient matrix structures. The shape functions for the polynomial order five are pictured in figure 1.

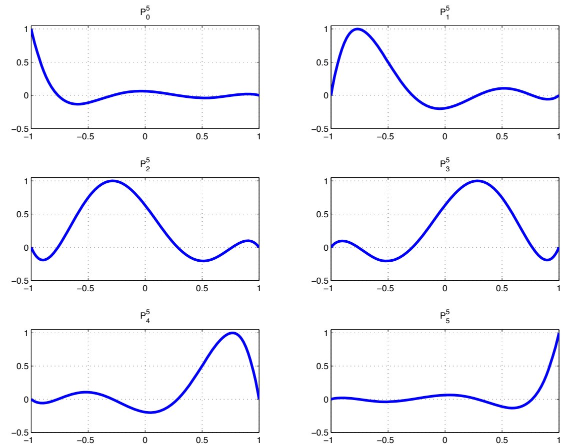

Figure 1: Lagrange interpolation polynomials of order 5, with supporting points at the zeros of the derivatives of $\mathcal{L}^5$

Mixed Variational Formulation

The underlying equations to the wave equation are the so called conservation equations for linear acoustics

$$\frac{1}{\rho_0 c^2}\frac{\partial p}{\partial t} + \nabla\cdot\vec{v} = F \,, \tag{2}$$

$$\rho_0 \frac{\partial \vec{v}}{\partial t} + \nabla p = 0 \,. \tag{3}$$

Here $\vec{v}$ denotes the acoustic velocity, p the pressure, and ρ_0 the density of the material. In order to obtain a stable finite element formulation, the unknowns are thought to be defined in different Sobolev spaces. This approach is called mixed variational formulation [2]. In the homogeneous form, this formulation reads as

$$\begin{aligned} \frac{\partial}{\partial t}\int_\Omega \frac{1}{\rho_0 c^2}\, p\, \varphi \,\mathrm{d}\vec{x} &= \int_\Omega \vec{v}\cdot\nabla\varphi \mathrm{d}\vec{x} - \int_{\partial\Omega} \varphi\, \vec{v}\cdot\vec{n}\mathrm{d}\vec{x} \\ \frac{\partial}{\partial t}\int_\Omega \rho_0 \vec{\psi}\, \vec{v}\,\mathrm{d}\vec{x} &= -\int_\Omega \nabla p\cdot\vec{\psi}\mathrm{d}\vec{x} \,. \end{aligned} \tag{4}$$

With $\vec{v},\ \vec{\psi} \in [L^2]^d$ and $p,\ \varphi \in H_0^1$. Here, d denotes the space dimension.

The spectral element discretization of the functional space of the unknowns is done according to [3]. In conclusion, we define $\mathbf{p} \in U_h^N$ and $\mathbf{v} \in V_h^N$, where

$$\begin{aligned} U_h^N &= \left\{\varphi_h \in H_0^1 \mid \varphi_h|_{K_j} \circ \mathbf{F}_j \in P_N\right\} \,, \\ V_h^N &= \left\{\boldsymbol{\psi}_h \in [L^2]^d \mid |\det \mathcal{J}_j|\mathcal{J}_j^{-1}\boldsymbol{\psi}|_{K_j} \circ \mathbf{F}_j \in [P_N]^d\right\} \,. \end{aligned} \tag{5}$$

This leads to the matrix system

$$\begin{pmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix} \begin{pmatrix} \dot{\mathbf{p}} \\ \dot{\mathbf{v}} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{R} \\ -\mathbf{R}^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ \mathbf{v} \end{pmatrix} \,, \tag{6}$$

where the element matrcies calculate as

$$\begin{aligned} \mathbf{D}^e &= [D_{ij}] = \omega_j \frac{1}{\rho_0 c^2}\, |\det \mathcal{J}_a|\, \delta_{ij} \,, \\ \mathbf{B}^e &= [\mathbf{B}_{ij}^{(d\times d)}] = \omega_j \rho_0 \frac{1}{|\det \mathcal{J}_a|} \mathcal{J}_a(\boldsymbol{\xi}_i)\mathcal{J}_a^T(\boldsymbol{\xi}_j)\delta_{ij} \,, \\ \mathbf{R}^e &= [\mathbf{R}_{ij}^{(1\times d)}] = \omega_j (\hat{\nabla}\hat{\varphi}_i(\boldsymbol{\xi}_j))^T \,. \end{aligned} \tag{7}$$

Here, $\mathcal{J}_a$ denotes the Jacobian matrix, ω_j the integration weight defined by the Gauss-Lobatto quadrature, d the space dimension and δ_{ij} the dirac delta. So in conclusion, $\mathbf{B}$ consists of $[d \times d]$ block diagonal entries, $\mathbf{D}$ is diagonal and $\mathbf{R}$ is sparse and consists of identical element matrices. It is thereby also possible to use explicite non-dissipative schemes without mass-lumping. For the following simulations, a leapfrog scattered time stepping algorithm is used as given in [4].

Numerical Results

As a first numerical example a one dimensional channel should be considered. The basic setup is pictured in figure 2. On the left-hand side, we apply a Gausian pressure pulse with a center frequency of $f_c = 3\text{kHz}$. The monitoring point is located $L_m = 23\lambda_c$ away from the source. The speed of sound in the domain is assumed to be $c = 340\frac{\text{m}}{\text{s}}$. For the spectral element simulations, the element size is chosen to be $L_e = 1.5\lambda_c$. Figure 3 pictures

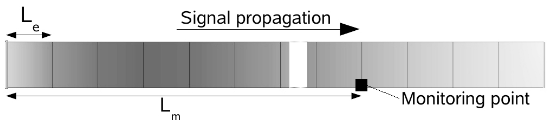

Figure 2: 1D channel setup

the signals received at the monitoring point. We have chosen a seventh order approximation for the spectral element mehtod and an equivalent discetization for linear finite elements ($\lambda_c/10$). Its visible, that the pulse gets distorted in the linear FE approximation. The spectral element simulation on the other hand, yields the correct shape.

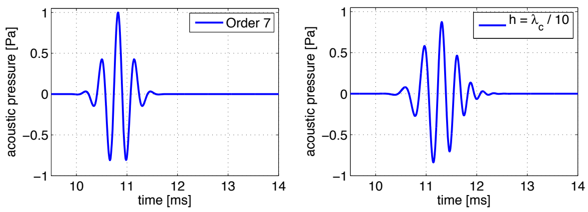

Figure 3: Signal at the monitoring point for spectral elements (left) and linear elements *h-FEM* (right)

In a second application, the behaviour of the method on distored meshes is investigated. The setup is defined as displayed in figure 4. At the source nodes, a single, 22kHz sine pulse is applied. The signal at the monitoring point should then be compared to the results of a simulation with a regular, very fine grid. Figure 5 shows the resulting signals for *spectral-* and *p-FEM*. One can see, that the third order spectral element approximation already yields good results which shows the robustness of the method against mesh distorsions.

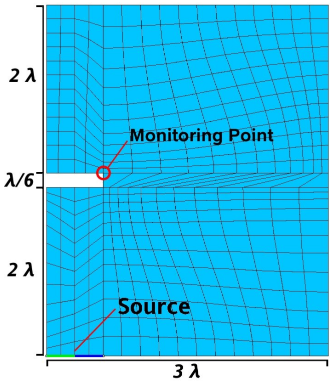

Figure 4: 2D setup, mesh distorsions

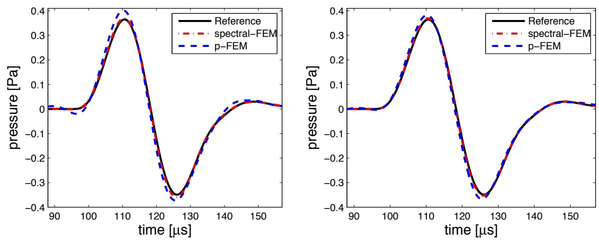

Figure 5: Signals at the monitoring point for approximation order three (left) and four (rght)

Summary

We have shown, that the method introduced, enables very robust and acurate simulations in the field of acoustics. Esspecially for large transient problemems which usually require a huge amount of memory. It should be emphasized, that the method presented here, directly applies to the governing equations of acoustics. This is a much more general approach and enables extensions e.g. in the field of aeroacoustics and for electromagnetic wave equations. In future work, this method and its possible extensions will be further investigated.

References

[1] C. Bernadi and Y. Maday. Spectral Methods. In *Handbook of Numerical Analysis, Vol. V.*

[2] D. Braess. *Finite Elemente.* Springer, Berlin, 2002.

[3] G. Cohen. *Higher-Order Numerical Methods for Transient Wave Equations.* Springer, Berlin, 2002.

[4] A. Hüppe. Spectral finite elements for wave equations. Master's thesis, Lehrstuhl für Sensorik der Friedrich-Alexander-Universität Erlangen-Nürnberg, November 2007.

[5] M. Kaltenbacher, A. Hauck, and R. Lerch. Hierarchische Finite-Elemente für die numerische Berechnung von akustischen Feldproblemen. In *Proceedings of the DAGA'07*, pages 207–208, 2007. CD-ROM-Proceedings, Stuttgart, 19.-22.03.2007.