Time Domain Boundary Conditions for Acoustic Conservation Equations

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Introduction

The treatment of free field as well as impedance boundary conditions is well known in the frequency domain. However, for time domain computations many problems arise such as the highly increased complexity of schemes as well as their stability. In the following we present approaches towards a perfectly matched layer (PML) formulation as well as a time domain impedance boundary condition (TDIBC) for computational acoustics. Both boundary conditions are implemented in a spectral element simulation scheme for the acoustic conservation equations in their mixed variational formulation.

Acoustic Conservation Equations

The conservation equations of acoustics, namely mass and momentum conservation can be written as

$$\frac{1}{\rho_0 c^2}\frac{\partial p'}{\partial t} + \nabla\cdot\vec{v}' = F \qquad (1)$$

$$\rho_0 \frac{\partial \vec{v}'}{\partial t} + \nabla p' = 0\ , \qquad (2)$$

where p' denotes the acoustic pressure, $\vec{v}'$ the acoustic particle velocity and F some source term. Furthermore, ρ_0 and c are the density and speed of sound in the medium of propagation.

Applying a finite element formulation to (1) and (2) as proposed in [1] leads not only to a stable and accurate but also to a very fast and efficient computational scheme ([2]).

Perfectly Matched Layer (PML)

PML techniques have been proven to show excellent absorption properties for waves at any angle of incidence. In this approach we follow [3] and introduce a complex coordinate stretching. The space derivative in x-direction is thereby written as

$$\frac{\partial}{\partial x} \rightarrow \frac{j\omega}{j\omega+\sigma_x}\frac{\partial}{\partial x}\ . \qquad (3)$$

Analogously, this is done for the other two space dimensions y and z. In the frequency domain, this change of variables can be directly applied to (1) and (2) thus obtaining the PML formulation.

In the time domain, this direct approach would give rise to a convolution integral which would make the scheme very complex and hard to handle. In a first step we investigate (2) and consider just the x-component

$$j\omega\ \rho_0 v_x' + \frac{j\omega}{j\omega+\sigma_x}\frac{\partial p'}{\partial x} = 0\ . \qquad (4)$$

By a multiplication with $(j\omega+\sigma_x)/j\omega$ we get a one dimensional PDE which can be easily transformed back into the time domain. By doing this for each space dimension separately, we obtain

$$j\omega\ \rho_0\vec{v}' + \rho_0\,[\sigma]\,\vec{v}' + \nabla p = 0\ ;\ [\sigma] = \mathrm{diag}(\sigma_x,\sigma_y,\sigma_z)\ . \qquad (5)$$

In a second step, we consider (1) in its frequency domain formulation

$$j\omega\ \frac{1}{\rho_0 c^2}p' + \frac{j\omega}{j\omega+\sigma_x}\frac{\partial v_x'}{\partial x} + \frac{j\omega}{j\omega+\sigma_y}\frac{\partial v_y'}{\partial y} + \frac{j\omega}{j\omega+\sigma_z}\frac{\partial v_z'}{\partial z} = 0\ . \qquad (6)$$

Now, we add to (6) the neutral terms

$$\pm\frac{\sigma_x}{j\omega+\sigma_x}\frac{\partial v_x'}{\partial x} \pm \frac{\sigma_y}{j\omega+\sigma_y}\frac{\partial v_y'}{\partial y} \pm \frac{\sigma_z}{j\omega+\sigma_z}\frac{\partial v_z'}{\partial z}\ . \qquad (7)$$

To enable an easy transformation back to the time domain, we introduce the vectorial auxiliary variable $\vec{q} = (q_x, q_y, q_z)$, whose x-component is defined as

$$q_x = \frac{1}{j\omega+\sigma_x}\frac{\partial v_x'}{\partial x}\ . \qquad (8)$$

Again y and z components are defined analogously. By transforming the system back into the time domain we finally arrive at the desired PML formulation

$$\frac{1}{\rho_0 c^2}\frac{\partial p'}{\partial t} + \nabla\cdot\vec{v}' + (\sigma_x,\sigma_y,\sigma_z)^T\cdot\vec{q} = 0 \qquad (9)$$

$$\rho_0\frac{\partial\vec{v}'}{\partial t} + \rho_0\,[\sigma]\,\vec{v}' + \nabla p = 0 \qquad (10)$$

$$\frac{\partial\vec{q}}{\partial t} + [\sigma]\,\vec{q} - \mathcal{B}\,\vec{v}' = 0\ , \qquad (11)$$

where $\mathcal{B} = \mathrm{diag}(\partial/\partial x, \partial/\partial y, \partial/\partial z)$.

The auxiliary variable $\vec{q}$ only exists in the PML region and can be defined in the space H^1.

As a test example, we choose the setup depicted in Fig. 1. On the lower left corner a sine-pulse Dirichlet condition for the acoustic pressure is applied. The speed of sound inside the domain is normalized to 1m/s. The damping parameter σ is chosen to increase within the PML layer by the function

$$\sigma(r) = \frac{c}{L-r}\ . \qquad (12)$$

In here, L denotes the width of the damping region. As seen in Fig. 2, the time signal at the monitoring point does not show any reflections. Thus showing that the proposed scheme works and provides stable and correct solutions.

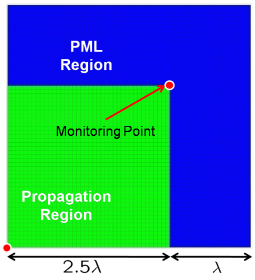

Figure 1: 2D Setup

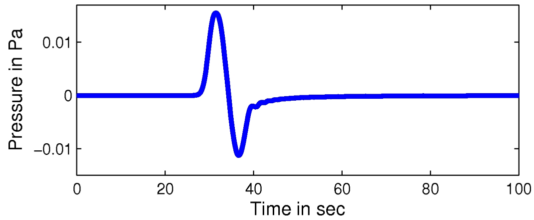

Figure 2: Time Signal of acoustic pressure at the monitoring point

Impedance Boundary Conditions

Acoustic pressure and acoustic particle velocity are related by the acoustic impedance Z_a

$$\vec{v}'(\omega) \cdot \vec{n} = \frac{1}{Z_a(\omega)} p'(\omega) = Y_a(\omega) p'(\omega) \; . \quad (13)$$

In the frequency domain, measured data can be incorporated directly into the scheme. In the time domain, on the other hand, one needs to model the impedance by a continuous function and evaluate a convolution integral.

Here, we follow [4] and choose the three parameter model

$$Y_a(\omega) = \frac{j\omega}{(j\omega)^2 M + j\omega D + K} \; . \quad (14)$$

In (14), M, D and K denote the damping parameters which have to be chosen according to a given design frequency. The corresponding time domain representation of (14) can be given as

$$y_a(t) = \frac{1}{2MC_2} \left(e^{z_1 t} z_1 - e^{z_2 t} z_2 \right) \; , \quad (15)$$

with $C_1 = D/(2M)$ and $C_2 = \sqrt{D^2 - 4\,K\,M}/(2M)$, as well as $z_1 = -C_1 + C_2$ and $z_2 = -C_1 - C_2$.

The time domain model is passive if the exponents z_1 and z_2 are negative, which gives rise to similar conditions as in [4]. The convolution integral, which appears during the transformation into the time domain, can be evaluated using the idea of recursive convolution

$$\vec{v}(n\Delta t) \cdot \vec{n} = \sum_{m=0}^{n-1} \int_{m\Delta t}^{(m+1)\Delta t} y(\tau) p(n\Delta t - \tau) \, \mathrm{d}\tau \; . \quad (16)$$

Assuming a piecewise linear approximation of the unknowns in time, we derive the following two accumulator variables

$$\begin{aligned} \Sigma_1(n\Delta t) &= p(n\Delta t) \frac{1}{2MC_2} \left(e^{z_1 \Delta t} - 1 \right) \\ &\quad + e^{z_1 \Delta t} \, \Sigma_1((n-1)\Delta t) \end{aligned} \quad (17)$$

$$\begin{aligned} \Sigma_2(n\Delta t) &= p(n\Delta t) \frac{1}{2MC_2} \left(1 - e^{z_2 \Delta t} \right) \\ &\quad + e^{z_2 \Delta t} \, \Sigma_2((n-1)\Delta t) \; . \end{aligned} \quad (18)$$

With these two variables, the acoustic particle velocity is set in each time step to

$$\vec{v}'(n\Delta t) \cdot \vec{n} = \Sigma_1(n\Delta t) + \Sigma_2(n\Delta t) \; . \quad (19)$$

Therewith, the complex evaluation of the convolution integral has been significantly reduced. The two accumulators have to be stored for every grid node on the boundary of the domain.

As a first test we model an impedance tube as used for measuring the absorption coefficient of a material ([5]). As an excitation, we apply a low pass filtered noise signal with frequency components from 0 to 1.6kHz to one end of the tube. On the other end, we apply the proposed impedance boundary condition. We choose a correct and stable set of parameters for the model and evaluate the absorption coefficient α of the acoustically treated end by the transfer function method. The results of the simulation is pictured in Fig. 3. It can be seen that the proposed scheme leads to stable results and the expected linear bahviour of the absorption coeffient can be obtained.

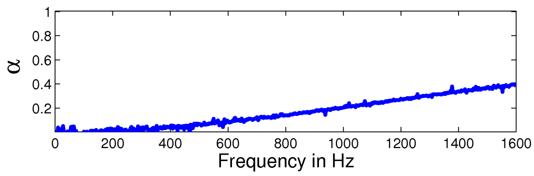

Figure 3: Obtained absorption coefficient

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