A Wave Based Method for solving Helmholtz problems in semi-infinite domains

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Introduction

The Wave Based Method (WBM) [1] is an alternative deterministic technique for the analysis of Helmholtz problems. The method is based on an indirect Trefftz approach [2], describing the dynamic response variables using wave functions which exactly satisfy the governing equation. This way, the method can avoid the fine discretisations and corresponding large systems needed by the element based methods to control the pollution errors [3], resulting in an enhanced numerical performance in an extended frequency range.

This paper discusses the application of the WB modelling process to a 2D or 3D semi-unbounded problem truncated by a rigid baffle plane. Important examples of this type are semi-infinite scattering and radiation problems and transmission problems.

The WBM for semi-unbounded problems

The WBM modelling procedure consists of the following three steps:

1. Partitioning into WBM subdomains

The semi-unbounded domain is separated into a bounded and an unbounded region by introducing a truncation surface Γ_t, which is either a hemicircle (for 2D problems) or hemisphere (for 3D problems). The interior is modelled using bounded subdomains, which must be convex to guarantee convergence [1]. The exterior, semi-unbounded part is modelled as one subdomain (figure 1.a). For transmission problems, the acoustic domain covers both sides of the rigid baffle. In this case, an additional independent unbounded subdomain is used, as shown in figure 1.b. These two semi-unbounded subdomains are coupled through the bounded part of the problem, inside the two truncations $\Gamma_{t,1}$ and $\Gamma_{t,2}$.

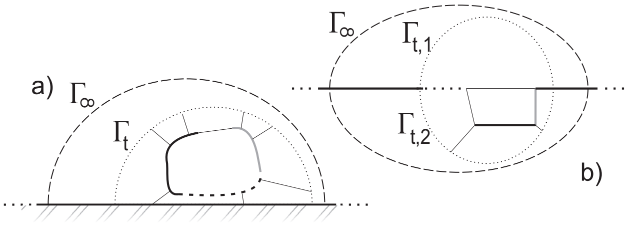

Figure 1: A WB partitioning of a semi-unbounded baffled (a) and transmission (b) problem

2. Acoustic pressure expansion

The steady-state acoustic pressure field $p^{(\alpha)}(\boldsymbol{r})$ in each (bounded or unbounded) acoustic subdomain $\Omega^{(\alpha)}$ is approximated by a solution expansion $\hat{p}^{(\alpha)}(\boldsymbol{r})$:

$$p^{(\alpha)}(\boldsymbol{r}) \simeq \hat{p}^{(\alpha)}(\boldsymbol{r}) = \sum_{w=1}^{n_w^{(\alpha)}} p_w{}^{(\alpha)} \Phi_w^{(\alpha)}(\boldsymbol{r}) + \hat{p}_q^{(\alpha)}(\boldsymbol{r}) \quad (1)$$

The degrees of freedom $p_w^{(\alpha)}$ are the weighting factors for each of the selected wave functions $\Phi_w^{(\alpha)}$. These functions are chosen to satisfy the Helmholtz equation a priori and are described in more detail by Desmet [1]. $\hat{p}_q^{(\alpha)}$ represents a particular solution resulting from an acoustic source term.

Wave functions for a semi-unbounded subdomain

The wave functions for a semi-unbounded domain are obtained as a subset of the functions for unbounded problems. The retained functions are selected such that they satisfy the velocity condition on the rigid baffle. For 2D problems, this yields [4]:

$$\Phi_w^{(\alpha,s-ub)}(\boldsymbol{r}\,(r,\theta)) = H_w^{(2)}(kr)\cos(w\theta), \quad (2)$$

with r and θ polar coordinates. $H_w^{(2)}(\bullet)$ is the w-th order Hankel function of the second kind.

In the case of a 3D problem, the functions are selected from the set proposed in [5], yielding:

$$\Phi_{wv}^{(\alpha,s-ub)}\left(\boldsymbol{r}(r,\theta,\phi)\right) = h_w^{(2)}(kr) Y_w^v(\theta,\phi)\ , \quad (3)$$

with $v = -w \cdots w$ and retaining only those functions where $w + v$ is even. $h_w^{(2)}(kr)$ is the w-th order spherical Hankel function of the second kind and $Y_w^v(\theta,\phi)$ are the spherical harmonics [5].

3. Assembly of the wave model and post-processing

Enforcing the boundary conditions and subdomain continuity through a weighted residual formulation yields a square matrix system, which can be solved for the unknown wave function contributions p_w [1]. The final step in the modelling process is the backsubstitution of these contribution factors into the pressure expansions (1), yielding an analytical description of the approximated dynamic pressure field.

2D numerical example

The WBM approach for modelling semi-unbounded problems is applied to analyse a piston of 1m wide, vibrating with unit velocity in a baffle plane. The WB truncation circle is chosen to exactly span the piston. One WB subdomain is used to model the bounded part. The WB result is validated using an analytic solution obtained by evaluating the Rayleigh integral formulation [6].

Figure 2 shows the relative pressure error convergence for an increasing number of degrees of freedom in the model, calculated along both a line through the origin and a circle centered on the origin. It can be clearly observed that the WB model converges to the analytical solution, and excellent accuracy is obtained using less than 100 functions.

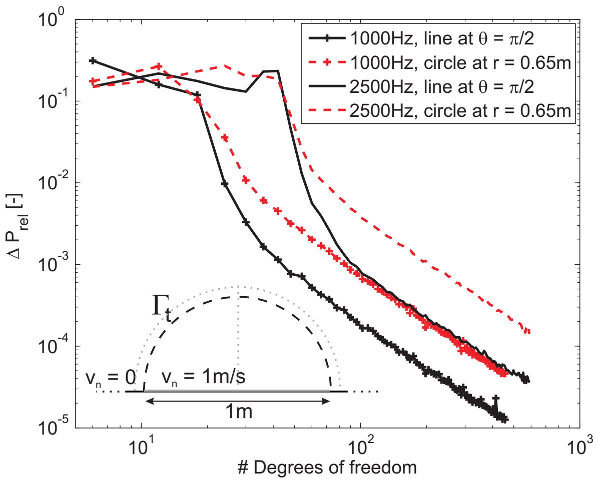

Figure 2: Relative pressure error convergence at 1000Hz and 2500Hz for a line at $\theta = \pi/2$ and a circle at $r = 0.65m$. Inset: problem definition

3D numerical example

This 3D case studies the transmission of a plane wave with propagation direction $(-1, 0, -2)$ trough a square cutout with edge length 1m in a baffle (XY coordinate plane). The WB model consists of two semi-unbounded subdomains and 12 bounded subdomains. Figure 3 compares the acoustic pressure field (amplitude) obtained with a WB model (left) versus a BE model (right), illustrating the good agreement between both results.

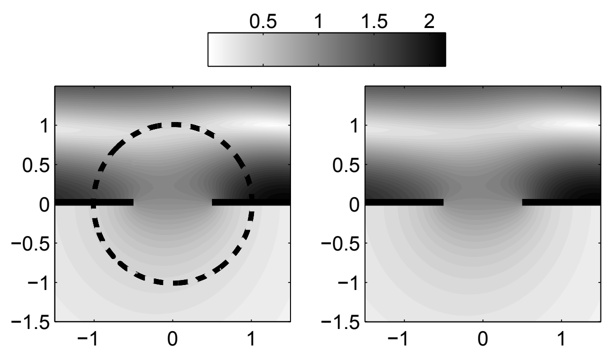

Figure 3: Pressure Amplitude [Pa] comparison, $100Hz$; left: WBM, right: BEM. The WB truncation sphere is indicated in dashed line (- -)

Figure 4 shows the active intensity at $100Hz$, illustrating that the field obtained using the reduced set (3) satisfies the baffle velocity condition. Derived quantities like intensity can be calculated without loss of accuracy by (analytically) deriving the wave-like basis functions.

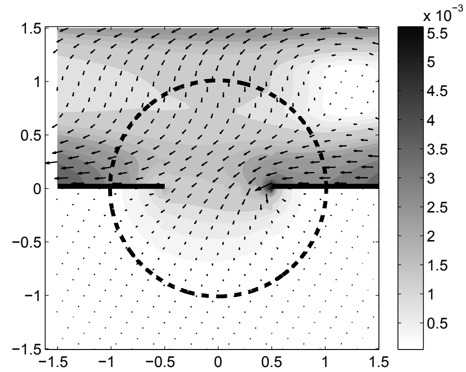

Figure 4: Active intensity in xz plane, $100Hz$

Conclusion

This paper discusses the use of the Wave Based Method for the treatment of both 2D and 3D Helmholtz problems in semi-infinite domains. To efficiently model those domains, a function set is proposed which not only satisfies the Helmholtz equation, but also the Sommerfeld radiation condition and the velocity condition on the rigid baffle plane. Two numerical validation examples illustrate the excellent accuracy and the computational efficiency of the proposed approach.

Acknowledgements

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