A Mixed Formulation for Computational Aeroacoustics applying Spectral Finite Elements

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Introduction

In the field of computational aeroacoustics (CAA) most computational schemes rely on a hybrid approach. From the results of a flow field computation, the appropriate source terms are computed by an aeroacoustic analogy and applied to the acoustic calculation scheme. Formulations which include effects of a mean flow on the acoustic wave propagation are mostly based on the linearized Euler equations approximated by finite difference or discontinuous Galerkin (DG) methods. In our contribution we apply a mixed variational formulation to the acoustic perturbation equations (APE) which include the effect of arbitrary mean flows on the acoustic wave propagation. The used finite element (FE) formulation has already been proven to provide a stable solutions to the conservation equations of acoustics [1] and shows high accuracy and computational efficiency [2].

Acoustic Perturbation Equations (APE)

The APE as well as the linearized Euler equations use a splitting of the unknowns velocity $\mathbf{u}$, pressure p and density ρ into mean and fluctuating parts. Additionally, in case of the APE, the fluctuating velocity terms are further split into their vortical and acoustical components

$$\begin{aligned} \mathbf{u} &= \overline{\mathbf{u}} + \mathbf{u}' = \overline{\mathbf{u}} + \mathbf{u^v} + \mathbf{u^a} \ , \\ p &= \overline{p} + p' \ , \\ \rho &= \overline{\rho} + \rho' \ . \end{aligned} \tag{1}$$

According to [3] the acoustic perturbation equations for a uniform mean flow $\overline{\mathbf{u}}_0$ can be given as

$$\begin{aligned} \frac{1}{\overline{c}^2\overline{\rho}}\frac{\partial p'}{\partial t} + \nabla\cdot\mathbf{u^a} + \overline{\mathbf{u}}_0\cdot\nabla p' &= q_c \ , \\ \overline{\rho}\frac{\partial \mathbf{u^a}}{\partial t} + (\overline{\mathbf{u}}_0\cdot\nabla)\mathbf{u^a} + \nabla p' &= \mathbf{q_m} \ , \end{aligned} \tag{2}$$

with $\overline{c}$ the speed of sound and q_c, $\mathbf{q_m}$ acoustic source terms. To ensure stability, a source filtering technique is applied which guarantees that only acoustic modes are excited by the sources. In the case of a vanishing mean flow ($\overline{\mathbf{u}}_0 = \mathbf{0}$) one obtains the conservation equations of linear acoustics

$$\begin{aligned} \frac{1}{c^2\overline{\rho}}\frac{\partial p'}{\partial t} + \nabla\cdot\mathbf{u^a} &= q_c \ , \\ \overline{\rho}\frac{\partial \mathbf{u^a}}{\partial t} + \nabla p' &= \mathbf{q_m} \ . \end{aligned} \tag{3}$$

They can be seen as a subset of the linearized Euler equations for isentropic media with no flow. Following [1]

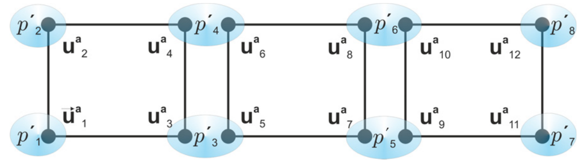

Figure 1: Location of unknowns in mixed formulation

one can give a variational formulation of (3) by using a mixed formulation in which the acoustic pressure p' is chosen from the continuous Sobolev space H^1 whereas the acoustic particle velocity $\mathbf{u^a}$ is approximated in the discontinuous Lebesque space L_2. The practical impact of this mixed approximation is depicted in Fig. 1 in which we see the location of unknowns for three finite elements. Additionally, in the discrete version, the acoustic particle velocity $\mathbf{u^a}_h$ is mapped from the grid element K to the reference element $\hat{K}$ with the $H(\mathbf{div})$ conforming Piola transformation

$$\mathbf{u^a}_h = \frac{1}{|\mathcal{J}_K|}\mathcal{J}_K\hat{\mathbf{u}}^{\mathbf{a}}_h \ . \tag{4}$$

The application of this scheme to the conservation equations has already shown the excellent properties in terms of accuracy and computational time [2].

A varaiational formulation of (2) can now be obtained by using a standard FE procedure in which we multiply the equations by appropriate test functions and perform an integration over the domain. A discretization with spectral finite elements then leads to the matrix system

$$\begin{pmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}\begin{pmatrix} \dot{\mathbf{p}}' \\ \dot{\mathbf{u}}^{\mathbf{a}} \end{pmatrix} + \begin{pmatrix} \mathbf{K_p} & -\mathbf{R} \\ \mathbf{R}^T & \mathbf{K_u} \end{pmatrix}\begin{pmatrix} \mathbf{p}' \\ \mathbf{u^a} \end{pmatrix} = \begin{pmatrix} Q_c \\ \mathbf{Q_m} \end{pmatrix} \ , \tag{5}$$

in which the matrices $\mathbf{D}$ and $\mathbf{B}$ are (block-) diagonal due to the spectral element discretization [4] thus enabling the usage of explicit time stepping schemes. Moreover, the matrix $\mathbf{R}$ does not contain any geometric information because of the Piola transformation of the velocity unknowns.

Acoustic pulse in uniform mean flow

To verify the approach, the benchmark problem depicted on the left hand side of Fig. 2 is considered [5]. As an initial condition at $t = 0$ we prescribe an acoustic pressure field given by

$$p'(t=0) = \exp\left(\ln(2)\frac{x^2 + (y+75)^2}{25}\right) \ ,$$

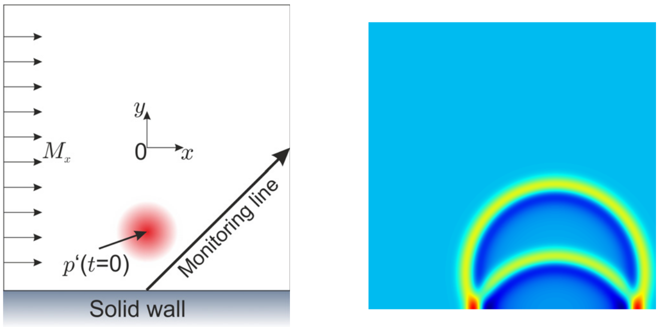

Figure 2: Scetch of computational setup (*left*) and computed acoustic pressure at T=60s (*right*)

while setting $\mathbf{u^a}(t = 0) = \mathbf{0}$. During the pulse propagation, we record the acoustic pressure for different time levels along the monitoring line plotted in Fig. 2. For the computation we choose a uniform flow velocity of $\overline{\mathbf{u}} = (0.5\overline{c}, 0)^T \rightarrow M_x = 0.5$. The domain is discretized by second order spectral elements with an edge length of $h = 1$m and a fourth order Runge-Kutta time stepping scheme is applied.

The pressure distribution along the monitoring line is given in Fig. 3 for $t = 20$s (left) and $t = 60$s (right). Analytical and the numerical solutions match well, which validates our approach.

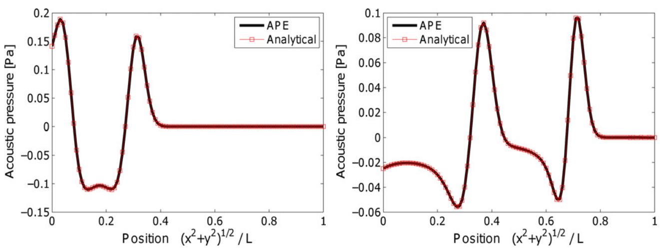

Figure 3: Pressure distribution on the monitoring line with length L for $t = 20$s (*left*) and $t = 60$s (*right*)

Acoustic source in shear flow

As a second example we investigate an acoustic monopole source in a sheared mean flow as given in Fig. 4 with $u_x = M_x \tanh(2y/\delta_w)$ and $\delta_w = 50$. The source is located at the center of the domain and is modeled by the term

$$q_c = 2\cos(\omega t)\exp\left(-\ln(2)\frac{x^2+y^2}{9}\right) .$$

Dimensions and discretization of the flow region are chosen as in the first test case. Additionally, surrounding the flow region, we define an attenuation region in which u_x is slowly decreased to reach zero at the interface to the perfectly matched layer (PML) [2]. Pictured on the right hand side of Fig. 4 are the contours of the acoustic pressure at time $t = 180$s. One can see, how the waveform is circular in the center and gets more and more distorted while reaching regions with higher flow velocities.

To investigate the additional computational effort, we choose a very large grid and compute the solution with

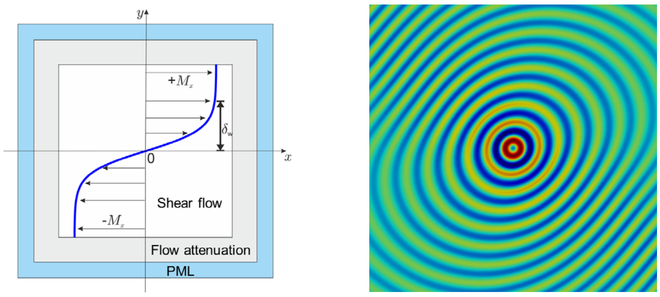

Figure 4: Scetch of computational setup (*left*) and computed acoustic pressure at $t = 180$s and $M_x = 0.3$ (*right*)

a sheared flow, i.e. system (2), and with zero flow, i.e. system (3). As it can be seen in Tab. 1, the memory consumption is increased because of the two additional bilinear forms whereas the computational time remains in the same range due to explicit time stepping.

Formulation	Memory consumption	CPU time
APE	10.1 GByte	1h 31min
Conservation Eqns.	5.7 GByte	1h 17min

Table 1: Computational costs with 8.66 million unknowns, 384400 elements, 600 time steps

Conclusion

We have presented the first results for the application of a mixed variational FE scheme to the acoustic perturbation equations. For the investigated Mach numbers, the scheme gives accurate solutions even for non-uniform flows. Nevertheless it has to be mentioned that the scheme shows instabilities when it comes to higher Mach numbers. Therefore, we will investigate an appropriate stabilization technique in a next step.

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