Time-Domain Translation Operators for the Fast-Multipole-Method

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Introduction

Acoustic simulation with the fast multipole method utilizes the scalar addition theorem of the spherical solutions of the Helmholtz equation, cf. [1, 3]. These solutions are used to decompose incident (1) and radiating (2) scalar wave fields

$$p(\boldsymbol{r}) = b_{nm}\, j_{nm}(k\boldsymbol{r}), \quad (1)$$
$$p(\boldsymbol{r}) = c_{nm}\, h_{nm}(k\boldsymbol{r}). \quad (2)$$

The above equation is in the frequency domain and uses Einstein's summing convention to obtain linearly combined basis solutions

$$j_{nm}(k\boldsymbol{r}) = j_{n'}(kr)\, Y_n^m(\varphi,\vartheta)\, \delta_{n'n}, \quad \text{and} \quad (3)$$
$$h_{nm}(k\boldsymbol{r}) = h_{n'}(kr)\, Y_n^m(\varphi,\vartheta)\, \delta_{n'n}, \quad (4)$$

using the expansion coefficients b_{nm} and c_{nm}. The sound pressure $p(\boldsymbol{r})$ is observed at $\boldsymbol{r}$, which can be expressed as radius r, zenith angle ϑ and azimuth angle φ in spherical coordinates, and k is the wave-number. The expansion involves spherical harmonics $Y_n^m(\varphi,\vartheta)$ and spherical Bessel functions $j_n(kr)$ for incident (regular), or spherical Hankel functions $h_n(kr)$ for radiating (singular) fields.

Scattering usually produces from the weighted incident elementary waves $b_{nm}j_{nm}(k\boldsymbol{r})$ a diffracting field $c_{nm}h_{nm}(\boldsymbol{r})$. Let us assume the scattering operation $g_{n'n}^{m'm}$ is known and yields a radiating from an irradiating field

$$c_{nm} = g_{n'n}^{m'm}\, b_{n'm'}. \quad (5)$$

Any scatterer can be described by a suitable $g_{n'n}^{m'm}$ in its local coordinate system. Given neighboring scatterers, the overall field results from the interaction of all scatterers, so-called multiple scattering. The interaction of individual scatterers requires coordinate transforms between the local coordinate systems.

Re-expansion at shifted coordinates.

Any basis solution can be re-expanded into basis solutions at shifted, local coordinates $\boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{d}$. In the frequency domain, this re-expansion uses the coefficients $w_{n'n}^{m'm}$ (translation operator)

$$j_{nm}(k\boldsymbol{r}) = w_{n'n}^{m'm}(\boldsymbol{d}, \|\boldsymbol{d}\|)\, j_{n'm'}(k\boldsymbol{r}'), \quad (6)$$
$$h_{nm}(k\boldsymbol{r}) = w_{n'n}^{m'm}(\boldsymbol{d}, \|\boldsymbol{r}'\|)\, e_{n'm'}(k\boldsymbol{r}'). \quad (7)$$

Two cases need to be addressed for the radiating solutions depending on whether the re-expansion yields a radiating or irradiating field at $\boldsymbol{r}'$

$$e_{n'm'}(\boldsymbol{r}') = \begin{cases} j_{n'm'}(k\boldsymbol{r}'), & \text{for } \|\boldsymbol{r}'\| < \|\boldsymbol{d}\|, \\ h_{n'm'}(k\boldsymbol{r}'), & \text{otherwise.} \end{cases} \quad (8)$$

Re-expansion coefficients are a linear combination of either spherical Hankel or Bessel functions

$$w_{n'n}^{m'm}(\boldsymbol{d},\alpha) = q_l(n',n,m',m,\boldsymbol{d}) \begin{cases} h_l(k\|\boldsymbol{d}\|), & \text{for } \alpha < \|\boldsymbol{d}\|, \\ j_l(k\|\boldsymbol{d}\|), & \text{for } \alpha = \|\boldsymbol{d}\|. \end{cases} \quad (9)$$

The goal of this contribution is about time-domain implementation of these re-expansion coefficients, based on the implementation of the spherical Hankel function, which has been given previously as discrete-time filters [2, 5, 6].

Hankel function based translation

In [4], a time-domain formulation is proposed based on differential operators. A straightforward implementation of the re-expansion coefficients as filter sets in time-domain is unfeasible because of extraordinary dynamic demands. This problem can be solved by a redefinition of the translation operation.

This is done by replacing expansion coefficients in (1) and (2) with spherical wave spectra $\phi_{nm|r_S}$ and $\psi_{nm|r_A}$. $\phi_{nm|r_S}$ is the spherical spectrum of a point source distribution at the radius r_S (causing irradiation) and $\psi_{nm|r_A}$ is the spherical spectrum of the pressure at r_A (denoting radiation)

$$b_{nm} = -i\,k\,h_{n'}(k\|\boldsymbol{r}_S\|)\, \phi_{nm|\boldsymbol{r}_S}\, \delta_{n'n}, \quad \text{and} \quad (10)$$
$$c_{nm} = h_{n'}^{-1}(k\|\boldsymbol{r}_A\|)\, \psi_{nm|\boldsymbol{r}_A}\, \delta_{n'n}. \quad (11)$$

The structure of $q_l(n',n,m',m,\boldsymbol{d})$ permits to combine the radial dependencies involved in (10) and (11) with the spherical functions involved in (9)

$$w_{c,n'n}^{m'm}(\boldsymbol{d},\boldsymbol{r}_A,\boldsymbol{r}_S,\|\boldsymbol{d}\|) = q_l(n',n,m',m,\boldsymbol{d}) \frac{h_l(k\|\boldsymbol{d}\|)}{-i\,k\,h_{n'}(k\|\boldsymbol{r}_A\|)\, h_n(k\|\boldsymbol{r}_S\|)}, \quad \text{for } \alpha \geq \|\boldsymbol{d}\|. \quad (12)$$

This limits the dynamic demands of a time-domain implementation: the maximum orders of the involved filters mainly determine the steepness of the frequency response. With the re-definition, the orders of the numerator and denominator are always equal, so that feasible frequency responses, cf. fig.1.

Moreover, the frequency responses of $w_{c,n'n}^{m'm}(\boldsymbol{d},\boldsymbol{r}_A,\boldsymbol{r}_S,\|\boldsymbol{d}\|)$ become now dependent on $\boldsymbol{r}_S$, $\boldsymbol{r}_A$, in addition to k and $\boldsymbol{d}$. Expanding the translation operator with radial steering filters (RSFs) on both sides, i.e. $h_n(kr_A)/h_n(kr'_A)$ and $h_n(kr'_S)/h_n(kr_S)$, the two dependencies r'_A and r'_S can be freely scaled

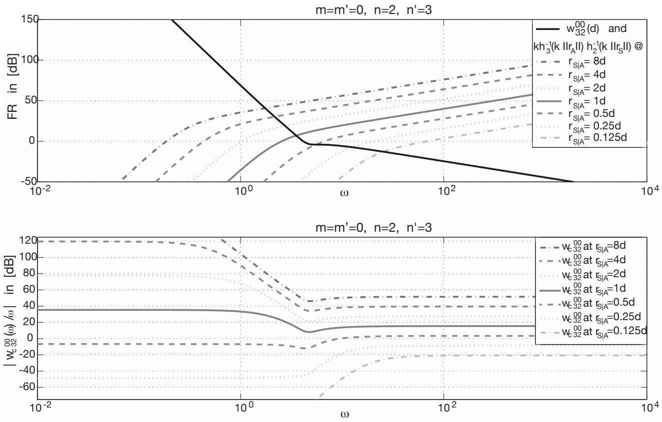

Figure 1: Frequency response of a typical Hankel based translation filter and the depending compensation filters caused by redefinition with different radii (top). Resulting freq. resp. of the depending compensated translation filter (bottom).

to achieve a numerically optimal implementation of $w_{c,n'n}^{m'm}(\boldsymbol{d}, \boldsymbol{r}'_A, \boldsymbol{r}'_S, \|\boldsymbol{d}\|)$. Numerical problems are hereby shifted to the RSFs where it is easier to stabilize them.

The result of this procedure are well-tempered filters for Hankel function based coordinate shifts as can be seen exemplarily in fig.1.

For the intended implementation as a discrete-time filter system, the modified re-expansion coefficients (12) can be expressed in the Laplace domain followed by a corrected impulse invariance transform.

Bessel function based translation

Implementing spherical Bessel functions as discrete time-domain filters causes numerical problems at small arguments. A possibility to realize these filters is the implementation as the real part of spherical Hankel filters. Based on recurrence relations for the Hankel filters [3] and their time reversed counterparts, which represents the depending complex conjugate, the Bessel filters can be implemented. The error of the discrete time-domain representation is minimized by employing a matched high pass filter of the order

$$n_{hpf} = 2(n-1), \tag{13}$$

where n denotes the order of the desired spherical Bessel filter. The cutoff frequency of the high pass filter is set to the frequency of the first root of the spherical Bessel function, fig.2.

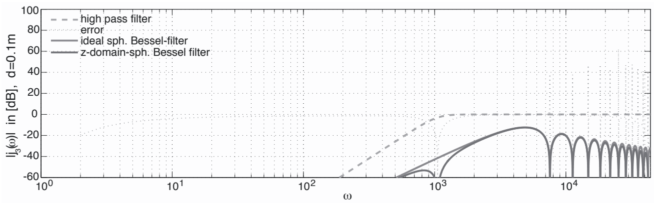

Figure 2: Typical frequency response of a spherical Bessel filter.

Example

Interaction of two scatterers.

To calculate the interaction of two scatterers at different coordinates $\boldsymbol{r}$ and $\boldsymbol{r}'$ due to the incident field at $\boldsymbol{r}$, the following equations are needed

$$c_{n'm'} = g_{nn'}^{mm'}\, b_{nm}\,, \tag{14}$$

$$c'_{n''m''} = g'^{m'm''}_{n'n''}\, w_{nn'}^{mm'}(\boldsymbol{d}, 0)\, b_{nm}. \tag{15}$$

Due to multiple scattering, the scattered field of both scatterers must include back scattering from the other scatterer

$$\begin{aligned} c_{n'''m'''} = {} & g_{nn'''}^{mm'''}\, b_{nm} + \\ & w_{n''n'''}^{m''m'''}(-\boldsymbol{d}, 0)\, g'^{m'm''}_{n'n''}\, w_{nn'}^{mm'}(\boldsymbol{d}, 0)\, c_{nm}\, \delta_{n'''n}^{m'''m}, \end{aligned} \tag{16}$$

$$\begin{aligned} c'_{n'''m'''} = {} & g'^{m'm'''}_{n'n'''}\, w_{nn'}^{mm'}(\boldsymbol{d}, \|\boldsymbol{d}\|)\, b_{nm} + \\ & w_{n''n'''}^{m''m'''}(\boldsymbol{d}, 0)\, g_{n'n''}^{m'm''}\, w_{nn'}^{mm'}(-\boldsymbol{d}, 0)\, c'_{nm}\, \delta_{n'''n}^{m'''m}. \end{aligned} \tag{17}$$

This creates two feedback loops that need to be solved for both radiating, scattered fields. Especially these feedback loops are predestined for a solution in time domain.

Conclusion

In a compact formulation we showed that translation operations for the fast multipole method can be realized as time-domain filters. Due to a redefinition of the translation operation Hankel filter based re-expansions can be realized. We also stated required solutions for the stabilization for Bessel filter based operations.

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