A solution of the Helmholtz equation with nonconforming finite elements on nonmatching grids

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Introduction

In the context of vibro-acoustic FE analysis the need of subdomain coupling is an integral part of the calculation process. The application of methods such as FETI or BETI assumes an independent subdomain modeling and discretization, which often results in nonmatching grids. A commonly used tool to ensure proper solution continuity through the subdomain interfaces is known as the Mortar technique. The standard Mortar technique imposes at the interfaces a condition of weak continuity on the solution. On the other hand,due to the relative sparse matrices one needs to handle during the implementation, the interest in the nonconforming finite elements stays constant during the last years.

In the current contribution a Mortar technique is presented for the solution of the Helmholtz equation discretized with nonconforming FE elements on nonmatching grids. The technique has been proven reliable for elliptic partial differential equations and holds a potential for delivering good results and decreasing the computation time. The aim of this work is to investigate the possible benefits from both independent meshing and nonconforming elements for the solution of the Helmholtz equation. The results from the application of the approach are compared to the results obtained for the nonmatching grids with standard conforming Lagrange elements and the further implementation of the studied technique for fluid-structure coupling is discussed.

The Helmholtz equation

In order to investigate the applicability of nonconforming finite elements for the solution of acoustic problems, respectively sound transmission loss problems, a simple example of a plane wave propagation in a fluid is initially taken into consideration. A fluid field, represented by a unit square domain, is excited at one edge by a time harmonic plane wave. On the opposite side of the domain an absorbing condition is introduced. The pressure based Helmholtz equation

$$\Delta p(x,y) + k^2 p(x,y) = 0 \quad \text{in} \quad x,y \in [0,1] \qquad (1)$$

is taken and the related boundary value problem (BVP) is defined. The plane wave source is determined as Neumann boundary condition

$$\nabla p(x,y) \cdot n = -i\omega\rho_0 V_n \quad \text{on} \quad \Gamma_s = 0 \times [0,1] \qquad (2)$$

$$i\omega\rho_0 C p + \nabla p(x,y) \cdot n = 0 \quad \text{on} \quad \Gamma_a = 1 \times [0,1] \qquad (3)$$

and the absorbing properties are introduced in the form of Robin type boundary condition. In the BVP formulation (1)-(3), k is the wave number, ω is the angular frequency, ρ_0 the fluid density, V_n the normal velocity and C the absorbing coefficient. With the assignment of constant normal velocity at all points on the boundary Γ_s, the problem can be reduced to one dimensional problem, which analytical solution is well known.

$$p(x) = -\frac{\omega\rho_0 V_n}{k} e^{-ikx} \qquad (4)$$

Neither for the analytical solution nor for the numerical approach damping is considered.

Nonconforming FE method

The finite element method is called nonconforming in the case when the trial functions form a space, which is not a subspace of the solution space[1]. In this contribution the term nonconforming is used only for the case of nonconforming elements. So-called P1 nonconforming elements are presented in particular, which are also known as lowest order Crouzeix-Raviart element depicted in Fig. 1a) for which the degrees of freedom are associated with the edge midpoints. Nonconformity resulting from nonmatching grids is denoted as a nonmatching grids case.

P1 nonconforming elements are the simplest nonconforming element type used for the solution of the second order elliptic boundary values problem. The relaxation of the strict conditions required by the conforming elements, sometimes also called “variational crimes”, give the nonconforming finite elements a certain number of advantages, as in the case of weak constraint enforcement (divergence-free flows in the Stokes problem), variable coefficient, curved boundaries and etc. [2]. Since the first introduction of the Crouzeix-Raviart elements for the solution of the Stokes problem in [3], these have been successfully implemented also for solving problems described by the Poisson equation and have been extensively further investigated. In this connection a significant amount of error analysis are performed, proving that the P1 nonconfoming elements maintain a convergence of the same order as the corresponding conforming elements. Based on the information about the flexibility and the good error estimates encountered by the application of nonconforming finite elements for the solution of second order elliptic partial differential equations, the implementation of the method for the solution of the Helmholtz equation has a good potential.

Nonmatching grids: Mortar method

In order to couple different domain discretization schemes as well as to tackle nonmatching grids at the subdo-

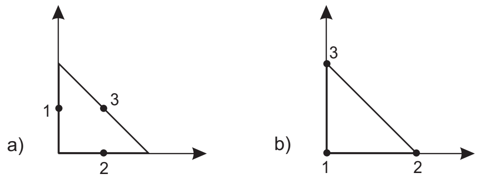

Figure 1: a) P1-nonconforming finite element (lowest order Crouzeix-Raviart finite element) b) P1-conforming finite element (Lagrangian finite element).

main interfaces the Mortar element method is used. This method is a domain decomposition technique with non-overlapping grids that enforces a weak continuity on the subdomain interfaces instead of pointwise continuity of the approximation functions. Since the application of mortaring conditions requests the function on the coupled interfaces to be known, the mortar methods in a case of P1 conforming and P1 nonconforming meshes differ.

Mortaring Crouzeix-Raviart FE

For the coupling of the nonmatching subdomain discretization with P1 nonconfoming elements, the mortaring technique presented in [4] is used. As the author suggests one side of the interface Γ_i, corresponding to the subdomain Ω_l, is chosen as a master $\gamma_{m,l}$ and the second one, corresponding to the subdomain Ω_k, as a slave $\delta_{m,k}$, see Fig. 2. The proposed technique requires that the trace of the solution of the two adjacent subdomains is L^2 projected on a trial mortar space and these two projections are equal. The mortar space should be determined by the slave mesh and defined on the common edge, here in particular a natural L^2 orthogonal basis is used. The test (mortar) space $M(\delta_{m,k})$ is introduced, as a subspace of $L^2(\Gamma_i)$, which consists of all piecewise constant function on the elements from the nonmortar (slave) triangulation on the interface. The dimension of the space $M(\delta_{m,k})$ is equal to the number of elements (midpoints) on the $\delta_{m,k}$. The L^2 orthogonal projection is $Q_m : L^2(\Gamma_i) \to M(\delta_{m,k})$, which is defined according to:

$$(Q_m p, v)_{L^2(\delta_{m,k})} = (p, v)_{L^2(\delta_{m,k})} \quad \text{for } \forall v \in M(\delta_{m,k}) \quad (5)$$

And the mortar condition is expressed by the equality of the projections of the traces onto the test space, represented for each interface by

$$Q_m p_k = Q_m p_l. \quad (6)$$

Now can be written the description of the discrete space V^h, which is not a subspace of the $H_0^1(\Omega)$ and consists of noncontinuous functions. That leads to a modification in the variational form of the discretized problem.

$$V^h = \{p_h \in X_h(\Omega) : \forall \delta_{m,k} = \gamma_{m,l} \subset \Gamma, Q_m p_k = Q_m p_l\} \quad (7)$$

Mortaring Lagrangian FE

In the case of standard Lagrangian FE the mortaring on the interfaces of the nonmatching grids have been done by means of a Lagrange multiplier technique. In the context of this approach a strong continuity is enforced for the flux through the boundary interfaces. One obtains:

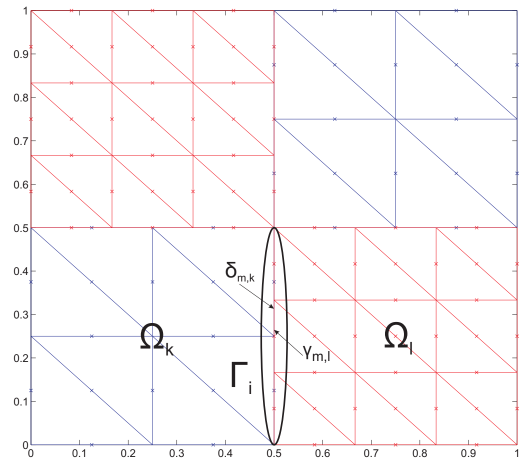

Figure 2: Nonmatching domain example with P1 nonconforming elements.

$$\lambda = -\frac{\partial p_k(\Gamma_i)}{\partial n} = -\frac{\partial p_l(\Gamma_i)}{\partial n} \quad \text{with} \quad \lambda \in M_h \quad (8)$$

Here M_h is a discrete Lagrange multiplier space. For the trace of the solution on the interfaces, an integral form of weak continuity is imposed:

$$\int_{\Gamma_i} (p_k - p_l)\mu d\Gamma \quad (9)$$

The test functions μ are chosen from a properly defined test function space. More details about the application of the method can be found in [5].

Solution

Complying with the mortaring conditions the matrix formulation of the problem in the cases of conforming (10) and nonconforming (11) finite element method are written:

$$\begin{pmatrix} M_c + i\omega C_c - \omega^2 K_c & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} \underline{p} \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} -i\omega \underline{G_c} \\ \underline{0} \end{pmatrix} \quad (10)$$

$$\left(M_n + i\omega C_n - \omega^2 K_n\right)\left(\underline{p}\right) = \left(-i\omega \underline{G_n}\right) \quad (11)$$

The mortar discretization of the conforming case initializes a saddle point structure of the linear system, which requires iterative solvers. For the solution of the linear system, arising from the nonconfoming mortar descretization, an additive Schwarz method is mostly recommended [4, 6, 7]. Taking into account the relatively small number of degrees of freedom of the investigated problem for the solution of both linear system of equations conjugate gradient method is used without preconditioners. The results achieved for the solution of the example problem on the nonmatching grids can be seen, for the real part of the solution in Fig. 3 and for the imaginary part in Fig. 4.

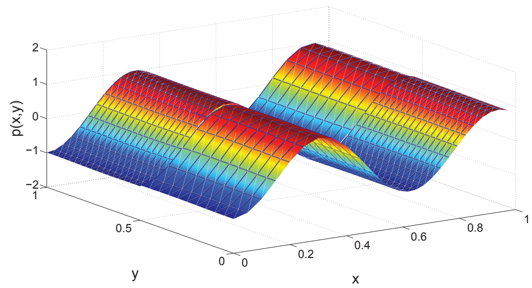

Figure 3: Real part of the solution.

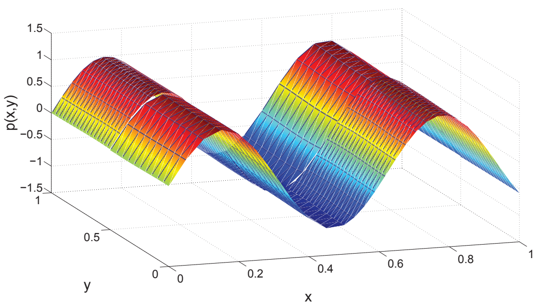

Figure 4: Imaginary part of the solution

Errors

Considering the error estimates with respect to the presented domain decomposition methods the investigated problem complies with the theoretical predictions. Proofs for the error estimates in connection with the mortaring techniques are available in [4] (for the nonconfoming mortaring technique) and in [5, 8] (for the application of the Lagrange multiplier based mortar method). In agreement with the previously performed studies and also to facilitate comparison, the discretizations errors are given in the L^2 norm. A special attention has been given to the analysis of the error between the results obtained from one side on matching grids and from the other side on nonmatching grids with mortar method applied. A comparison has been made in a case of domain discretized with Lagrange FE and with Crouzeix-Raviart FE. The errors with respect to the degrees of freedom have been analyzed and the results can be seen in Fig. 5. As it can be concluded from this figure, with the increase of degrees of freedom the error convergence rate for the both finite element types follow a similar path. For profounder error analyses, as well as more examples one can refer to [9].

Conclusions and Outlook

The proposed approach for the solution of the pressure based Helmholtz equation confirmed the potential suitability of the nonconforming finite elements method in acoustics. Considering the achieved results, it should be pointed out that dealing with nonmatching grids in this context does not require additional unknowns i.e. Lagrange multipliers. Mortaring technique used for the P1 nonconforming domain decomposition method gath-

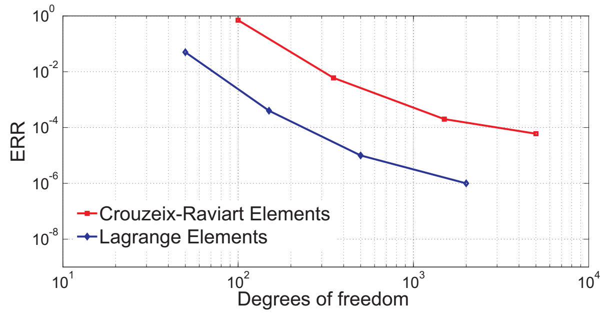

Figure 5: Comparison of the convergence performance for Mortar methods for different finite element types.

ers information from the entire element discretization at the coupled interfaces, which could be a drawback with respect to computation time. But on the other hand new mortaring techniques are already available, which handle this problem and restrict the necessary information to the interface nodes without compromising the precision. The well known iterative solvers, as the one based on conjugate gradient method, can be used for the solutions of the resulting linear systems of equation, as only a different type of preconditioners can be required. The Crouzeix-Raviart finte elements have been proven to deliver satisfactory results after enrichment also for the solution of the equation of motion, which opens interesting possibilities for the realisation of elasto-acoustic couplings. As it has been suggested in [10] fluid-structure interaction could be modeled also with the use of a combination between Crouzeix-Raviart and Raviart-Thomas finite elements based on the displacement-displacement formulation. In such a case the spurious modes free solution of the displacement based Helmholtz equation, achieved with the help Raviart-Thomas element domain discretization, could be coupled with enriched Crouzeix-Raviart elements used for the solid. Therefore the nonmatching grids situation together with the above mentioned coupling is a quite promising area of investigations.

In the context of transmission loss calculation of multilayered structures the use of independent domain discretization techniques and an optimal coupling method between the structures and fluids have always been an issue.

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