Adjoint Based Data Assimilation of Sound Sources

Mathias Lemke[1], Christian Westphal[1], Julius Reiß[1], Jörn Sesterhenn[1]

[1] *Institut für Strömungsmechanik und Technische Akustik, 10623 Berlin, Deutschland, Email: mlemke@tnt.tu-berlin.de*

Introduction

Adjoint equations have attracted great attention in the field of fluid dynamics as they provide optimal changes of flow configurations in a computational efficient manner. Their application covers optimisation, data assimilation, active control, sensitivity analysis and model reduction purposes.

In data assimilation adjoint equations are used to adapt parameters of a model to measurements. The difference between a computed state and a measurement is to be minimised, which defines a so called objective function. In this work the adjoint approach is used to identify sound source positions and signals from discrete microphone measurements. The Euler equations are considered as the governing model. As assimilation parameters sources for mass and momentum are used. The derived framework is validated by means of a synthetic configuration as well as an experiment.

In addition to the source identification an adjoint based approach to assess acoustic measurement configurations is presented.

Adjoint Approach

Following [1] adjoint equations arise by the so called objective function

$$J = g^T q, \tag{1}$$

defined as the product of a weighting g and the system state q. The state q is given as solution of equation

$$Aq = f, \tag{2}$$

representing the considered model, governed by e.g. the linearised Euler equations. The term f on the right hand side acts as source and is used as control parameter. To modify J by means of f the linear system (2) is to solved for every mentioned f.

The use of the adjoint equation

$$A^T q^* = g \tag{3}$$

can reduce the computational effort as the following holds

$$J = g^T q = (A^T q^*)^T q = {q^*}^T A q = {q^*}^T f. \tag{4}$$

The objective for different f can be computed by a cheap scalar product, once the adjoint equation is solved. Thus it allows an efficient determination of the impact of different f. For non linear problems an analogue relation can be found

$$\delta J = g^T \delta q = {q^*}^T \delta f. \tag{5}$$

Adjoint Euler Equations

To follow the aforementioned approach the governing equations have to be linearised. In context of the indented application the Euler equations are used in the following formulation.

$$\partial_t \begin{pmatrix} \varrho \\ \varrho u_j \\ \frac{p}{\gamma-1} \end{pmatrix} + \partial_{x_i} \begin{pmatrix} \varrho u_i \\ \varrho u_i u_j + p\delta_{ij} \\ \frac{u_i p \gamma}{\gamma-1} \end{pmatrix} - u_i \partial_{x_i} \begin{pmatrix} 0 \\ 0 \\ p \end{pmatrix} = f \tag{6}$$

Within ϱ denotes the density, u_j the velocity in x_j-direction and p the pressure. The energy equation is formulated in terms of p assuming a constant heat capacity. γ denotes the adiabatic coefficient. The right hand side f corresponds again to source terms, e.g. mass- and momentum sources. For sake of clarity the equations are abbreviated as

$$\partial_t a + \partial_{x_i} b^i + C^i \partial_{x_i} c = f. \tag{7}$$

Linearisation with respect to the actual system state $q = [\varrho, u_j, p]$ lead to

$$\partial_t \underbrace{\frac{\partial a_\alpha}{\partial q_\beta}}_{A} \delta q_\beta + \partial_{x_i} \underbrace{\frac{\partial b_\alpha}{\partial q_\beta}}_{B^i} \delta q_\beta + C^i \partial_{x_i} \delta q_\beta + \delta C^i \partial_{x_i} c_\beta = \delta f. \tag{8}$$

To derive the adjoint the linearised equations are used as additional constraint and added to the objective function in a Lagrangian manner.

$$\begin{aligned} \delta J &= g^T \delta q \\ &\quad - {q^*}^T \underbrace{\left(\partial_t A \delta q + \partial_{x_i} B^i \delta q + C^i \partial_{x_i} \delta q + \delta C^i \partial_{x_i} c - \delta f\right)}_{=0} \end{aligned} \tag{9}$$

The relation holds in an integral sense $\iint \mathrm{d}\Omega$ with Ω as space-time measure. However, for sake of clarity the integrals are not shown. Integration by parts lead to a formulation for δJ, that is independent from δq as in the example before.

$$\begin{aligned} \delta J &= \delta q^T g \\ &\quad + \delta q^T A^T \partial_t q^* + \delta q^T {B^i}^T \partial_{x_i} q^* \\ &\quad + \delta q^T \partial_{x_i} {C^i}^T q^* - \delta q^T \tilde{C}^i \partial_{x_i} c + {q^*}^T \delta f. \end{aligned} \tag{10}$$

Time wise and spatial boundary condition arising from the integration are neglected for clarity. These would give arise to the adjoint boundary and initial conditions. The operator $\tilde{C}^i$ abbreviates

$$q^*_\alpha \delta C^i_{\alpha\beta} \partial_{x_i} c_\beta = q^*_\alpha \delta q_\kappa \frac{\partial C^i_{\alpha\beta}}{\partial q_\kappa} \partial_{x_i} c_\beta. \tag{11}$$

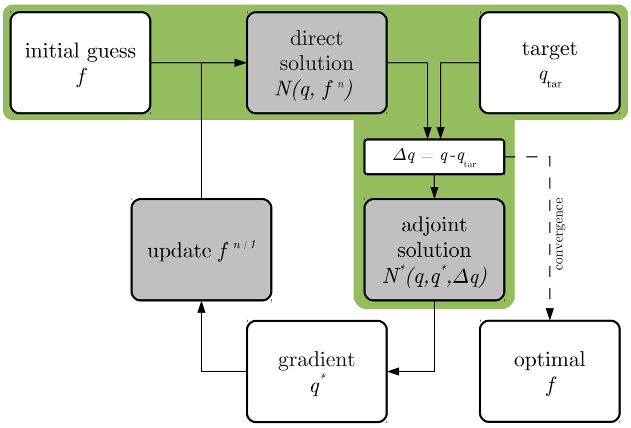

Figure 1: Iterative data assimilation loop. Only the tasks highlighted in green are necessary for sound source identification. Expansive operatives are marked by a grey box.

Collecting all terms linear in δq and choosing q^* to fulfil

$$\partial_t q^* = -{A^T}^{-1} {B^i}^T \partial_{x_i} q^* - {A^T}^{-1} \partial_{x_i} {C^i}^T q^* + {A^T}^{-1} \tilde{C}^i \partial_{x_i} c - {A^T}^{-1} g, \qquad (12)$$

leads to the change of the objective function

$$\delta J = {q^*}^T \delta f \qquad (13)$$

in analogue to the introductory example before. Thus the adjoint equations provide sensitivity information how the objective is changed by modification of forcing f. The obtained information can also be interpreted as gradient

$$\nabla_f J \approx q^*, \qquad (14)$$

which holds for infinitesimal small changes. In general the approach is applied in an iterative manner, see Fig. 1. The forcing f is adapted until the objective function is sufficiently small. Convergence acceleration methods like line search or use of non-linear conjugated gradients can improve the performance.

Pressure Based Objective Function

For acoustic problems the objective function is formulated in terms of pressure.

$$J = \frac{1}{2} \iint\limits_{\mathrm{d}\Omega} (p - p_{\exp})^2 \, \sigma_{x_i} \sigma_t \mathrm{d}\Omega \qquad (15)$$

The integral difference between the actual numerical state p and the measurement $p_{\exp}$ is used. Therein the weightings σ correspond to spatial and temporal weighting functions. These are equal to one if a measurement is available and zero if not. Discrete measurements are approximated by a sharp Gaussian distribution in order to avoid discrete excitation of the equations and therefore potentially unstable computations. In the time domain a corresponding treatment is found not to be necessary. Thus the variational change of J with respect to the system state is defined by

$$\delta J = \underbrace{(p - p_{\exp}) \, \sigma_{x_i} \sigma_t}_{g^T} \, \delta p. \qquad (16)$$

Again the integrals are omitted.

Potential Source Position and Signal

As acoustic sources can be described by mass, momentum and energy sources, they can be assimilated by means of the aforementioned adjoint approach. In general the iterative procedure may be applied to find an optimal excitation leading to a solution matching to the measurements.

However, this is not necessary as the first adjoint solution contains sufficient information for characterisation of the source in potential position and signal. By pointwise summation of the absolute adjoint sensitivities s over all computed time steps

$$s = \sum_{t_0}^{t_{n=\mathrm{end}}} \|q^*\|, \qquad (17)$$

the positions of maximum impact to the objective function are identified. If $f^0 =$ is applied, these correspond to the most likely source positions. Where the summarised sensitivity remains zero no source is located or at least cannot be captured by the measurements.

Once a position is identified the source signal can be obtained. The adjoint solution contains the information how to modify the initial guess $f^0 = 0$. Thus it contains also the signal in phase, here for the mentioned acoustic system. If also the amplitude is needed it can be obtained by application of the iterative framework.

Adjoint Based Observability Analysis

The adjoint equations provide the possibility to assess measurement configurations by analysis of the observability Gramian. For a real valued problem it is defined as

$$W_o = \int_0^\infty e^{N^T t} C^T C e^{N t} \, \mathrm{d}t \qquad (18)$$

in the context of a standard input-output system

$$\begin{aligned} \dot{q} &= Nq + Bf \\ y &= Cq \end{aligned} \qquad (19)$$

with q as state vector, f as input, y as output, N as governing model operator and B and C as weightings for input (forcing) and output (measurement). The Gramian measures the degree each system state influence the output. A direct computation of the Gramian is not suitable, but an approximation can be used. The so called empirical Gramian bases on impulse responses of the adjoint of (19).

$$W_o = \int_0^\infty q_i^* {q_i^*}^T \, \mathrm{d}t \qquad (20)$$

This integral can be approximated by means of adjoint snapshots, see [4] for details.

$$W_o = YY^T \qquad (21)$$

Therein Y represents m snapshots of the adjoint system for k measurement locations

$$Y = [\mathbf{q_1^*}(t_1) \ldots \mathbf{q_1^*}(t_m) \ldots \mathbf{q_k^*}(t_1) \ldots \mathbf{q_k^*}(t_m)]. \qquad (22)$$

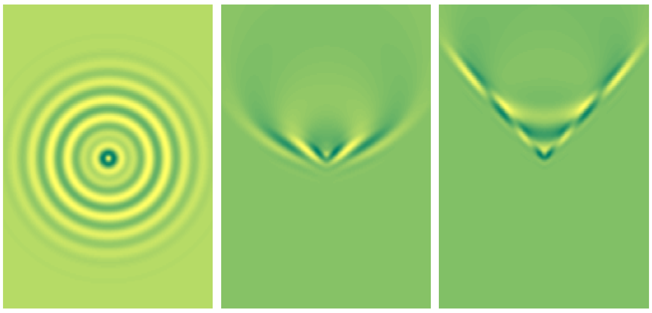

Figure 2: Exemplary observability modes for a single microphone in an inflow (from top to bottom). Different flow speeds Ma=0.0, 1.0, 1.5 (from left to right) are examined.

A modal decomposition of the resulting operator give rise to the (most) observable states. To reduce the computational effort a singular value decomposition is used to obtain the modes. Due to the structure of W_o the singular vectors correspond to the eigenvectors as

$$W_o = YY^T = U\Sigma VV^T\Sigma U^T = U\Sigma^2 U^T \quad (23)$$

holds. The applicability of the linear approach to nonlinear problems is shown in [2].

For sake of clarity the resulting observability modes of one microphone in a plane inflow are shown, exemplary for mode number four. The results base on two-dimensional simulations of the adjoint Euler equations. For approximation of the observability Gramian 300 time steps are used.

For a zero mean flow the resulting modes are circular. All pressure disturbances are equally observable. No directivity is found as only a single microphone is used. In case of a sonic inflow the resulting modes contain no parts downstream of the measurement. This is plausible as no acoustic signal created there can reach the microphone. For the supersonic case at Ma=1.5 a Mach cone like observability is found. The cone is characterised by an angle of about 42° according to a travelling source with corresponding velocity.

In general the resulting modes can be used to quantify the observability of certain structures. The projection of it onto the obtained modes give rise to the most observable parts. By subsequent subtraction the non-observable part arise. An according norm of the difference can be used for quantification of the observability. Thus a framework is available to optimise either measurement configurations to be most sensitive with respect to certain signals or to optimise signals to be most observable for a given measurement configuration. The application to a complex flow configuration is shown in [3].

However, this manuscript focuses on the identification of sound sources and signals. The observability analysis is not pursued further.

Synthetic Example

To validate the aforementioned adjoint framework a numerical test case is carried out. The mentioned computational domain covers a rectangular domain with an extent of $0.8 \times 0.8 \times 0.0875\text{m}^3$ and is discretised with $256 \times 256 \times 16$ points. On all boundaries characteristic type non-reflecting boundary conditions are used. For damping of spurious reflections a thin sponge layer region is applied. The test aims at identification of a harmonic source by eight surrounding microphones. Source and microphones are located in a common plane, see Tbl. 1 and Fig. 3 for details.

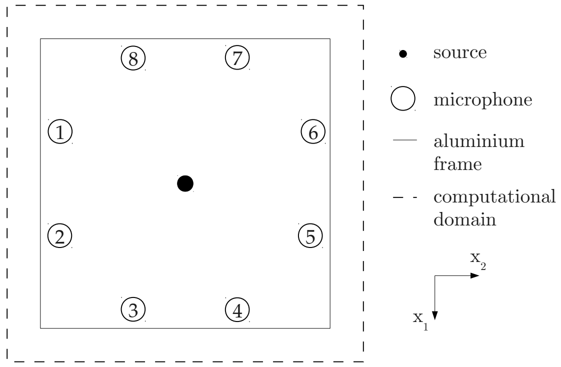

Figure 3: Sketch of the computational and experimental validation configuration.

mic	1	2	3	4	5	6	7	8
x_1 [mm]	233	467	632	633	467	233	67	68
x_2 [mm]	69	68	233	467	631	637	467	233

Table 1: Microphone positions for the synthetic and the experimental validation.

In the first step the reference sound field is computed. Therefore the mentioned Euler equations are excited in the pressure equation. The harmonic excitation with a frequency of 5kHz is centered in the domain. Its spatial distribution corresponds to a Gaussian distribution. The resulting pressure signals at the microphone positions are saved.

For the assimilation the reference excitation is disabled and the Euler equations are solved again forward in time. As no excitation is present the system state remains unchanged for all computed time steps. Base on this solution the adjoint equations are solved backwards in time excited by the difference between the reference signal and the actual solution. The excitation is controlled by the weighting σ_{x_i} corresponding to the microphone positions.

The normalised accumulated sensitivities (17) for the quantity related to pressure p are shown in Fig. 4. A symmetric result is found. The maximum sensitivity is centered in the computational domain and corresponds to the reference source position. Also the shape/monopol type of the excitation is recovered. Occurring side maxima are caused by interference effects and correspond to alternative source positions. A unique position can not be identified. For non harmonic signals the solution might be clearer.

Once the position is identified the signal can be extracted

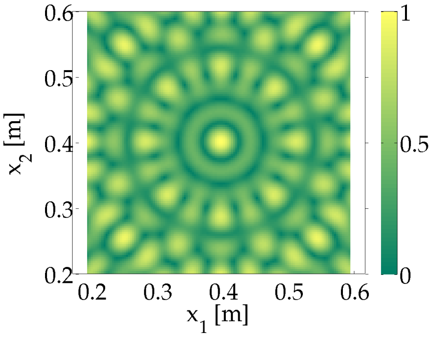

Figure 4: Resulting sensitivity field for the synthetic case. The maximum amplitude is found at the source position.

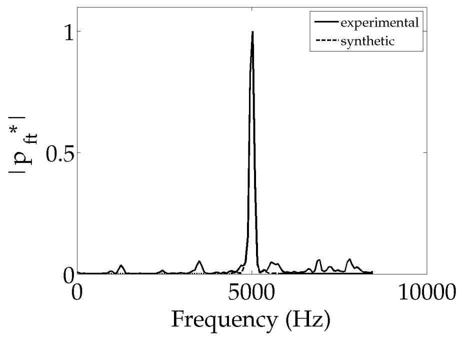

Figure 5: Frequency analysis of the adjoint signal for the synthetic and experimental case. For both the reference signal is recovered.

from the adjoint solution. A frequency analysis of the adjoint signal p^* shows, that the harmonic signal is recovered, see Fig. 5. However the amplitude of the source remains unclear. To obtain it the adjoint framework has to be applied in an iterative manner, what is not discussed here.

Experimental Validation

To show the applicability of the derived framework also an experimental validation is carried out. The experiment corresponds to the synthetic case before and is carried out in the anechoic room facility of Technische Universität Berlin. As source a small speaker mounted on a rope is used. For the measurements eight pre-polarised condenser microphones with a sampling rate of 48kHz are used. The weighting function σ_t is chosen accordingly. To avoid unwanted reflections, not captured by the applied model the supporting structure is covered with acoustic foam.

The resulting sensitivities are shown in Fig. 6. Again the centre is found to be a potential source location. Also the analysis of the adjoint signal correspond to the reference signal of 5kHz.

However, the maximum of the sensitivity occurs near the microphones 4 and 5. It corresponds to a side maxima in the synthetic case. A further analysis shows that the increased sensitivity is caused by the orientation of the speaker. The speaker is no monopol. Application of the iterative procedure shows, that mass- and momentum sources have to be applied to recover the microphone signals.

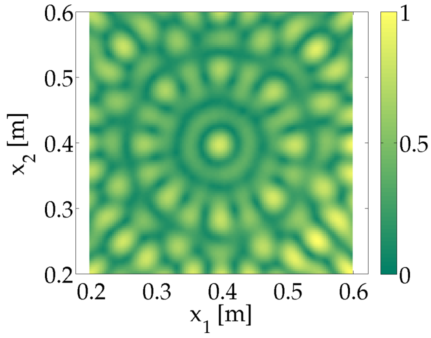

Figure 6: Resulting sensitivity field for the experimental case. The maximum amplitude is found near the microphones 4 and 5 due to the orientation of the used speaker.

Conclusion

A framework for identification of sound sources is presented. Based on adjoint sensitivities potential source locations can be obtained as well as the corresponding signal. The framework allows the identification without knowledge of the source type. Rather it provides corresponding information. A synthetic and an experimental validation are carried out showing the applicability. Furthermore an adjoint based method is presented to assess measurement configurations by means of an adjoint based approximation of the observability Gramian.

Acknowledgement

The authors gratefully acknowledge support by the Deutsche Forschungsgemeinschaft (DFG) as part of collaborative research center SFB 1029 "Substantial efficiency increase in gas turbines through direct use of coupled unsteady combustion and flow dynamics".

References

[1] M. Giles and N. Pierce. An introduction to the adjoint approach to design. *Flow, Turbulence and Combustion*, 65:393–415, 2000.

[2] S. Lall, J. E. Marsden, and S. Glavaški. A subspace approach to balanced truncation for model reduction of nonlinear control systems. *International Journal of Robust and Nonlinear Control*, 12(6):519–535, 2002.

[3] M. Lemke, J. Schulze, and J. Sesterhenn. Adjoint-based reconstruction of an entropy source by discrete temperature measurements. *Int. J. Computational Science and Engineering*, 9(5/6):526–537, 2014.

[4] C. W. Rowley. Model reduction for fluids, using balanced proper orthogonal decomposition. *Aerospace Engineering*, 15(3):997–1013, 2005.