Physical Modelling of Guitar String with Realistic Boundary Conditions

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The physical modelling of guitar strings consists of two main tasks: First, the modelling of an interior model describing the oscillation behavior of the string itself and, second, the incorporation of realistic boundary conditions connecting the string to a guitar body. For the modelling of the interior string model the Functional Transformation Method can be used, which turns a physical system described by a set of partial differential equations into a synthesis algorithm with real-time capability. In a second step realistic boundary conditions can be realized by the design of a feedback loop around the interior model.

Introduction

The vibrations of guitar strings are a well studied subject. Based on the fundamental laws of elasticity, their dynamic behavior is accurately described by differential equations of different kinds, see [1] for further references. There are several methods to turn these partial differential equations into physical simulation models by discretization in space and time or by functional transformations [2, 3].
Beside the vibration of the string itself, also the connection to other parts of the guitar have to be investigated to achieve a realistic sounding physical model. But the complexity of the boundary conditions highly scales the complexity of the initial-boundary-value problem.
A suitable modelling technique is the application of functional transformations, which are based on a modal expansion of an initial-boundary-value problem [4]. These techniques lead to powerful physical models, which can be formulated in a state-space description. Additionally they allow to adjust their boundary behavior for the realization of realistic boundary conditions.

String Oscillation

The oscillation of a single guitar string is a combination of spatial and temporal oscillations, and is not just a single oscillation with the fundamental frequency. The oscillation is shaped by several harmonics, which form the typical sound of a string.
Fig. 1 shows the shape of a single string oscillation (top) and the decomposition into its harmonics (bottom). Each harmonic has a different spatial wavelength λ and time period T. Also the amplitude parameters b are different. The varying wavelengths and periods of the harmonics lead to different propagation velocities. In general: The higher the frequency of a harmonic, the higher the propagation speed. This effect is known as dispersion and is typical for the sound of guitar strings.

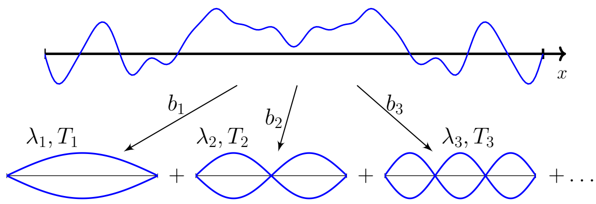

Figure 1: Guitar string oscillation decomposed into its harmonics: b_n: Amplitude parameter, λ_n: spatial wavelength, T_n: time period, x: space coordinate.

Beside the oscillations on the string, the boundary conditions are important for a realistic sound. Typically, a string is oscillating in a complex scenario and is interacting e.g. with the bridge, saddle, fingers, etc.

Modelling Approach

In the following sections a physical string model with realistic boundary conditions is derived in the continuous frequency domain. The modelling process follows the steps in Fig. 2. At first, a physical description of the guitar string is derived. This physical description is transformed by functional transformations into a physical model in terms of a multi-dimensional transfer function. This general model first assumes a simplified set of boundary conditions. Subsequently, a set of more realistic conditions is incorporated into the model.

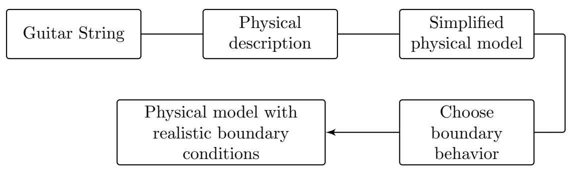

Figure 2: Individual steps of the modelling approach.

Physical Description

The scenario of a vibrating string on a guitar can be described physically by an initial-boundary-value problem: A partial differential equation, describing the oscillation of a guitar string, a set of boundary conditions characterizes the behavior of the ends of the string and the initial conditions describe the behavior of the string at time $t = 0$.

Partial Differential Equation

The propagation of transversal waves on a single guitar string can be described by a PDE in terms of the string deflection [1]. It depends on the position on the string x and time t ($y = y(x,t)$). $\dot{y}$ represents the time- and y' the space-derivative

$$\rho A \ddot{y} + EI y'''' - T_\mathrm{s} y'' + d_1 \dot{y} - d_3 \dot{y}'' = f_\mathrm{e}, \tag{1}$$

with the cross-section area A, area moment of inertia I and the length ℓ. The material is characterized by the density ρ and Young's modulus E. T_s describes the tension and d_1 and d_3 are constant and frequency dependent damping [1, 2]. The excitation function is defined as $f_\mathrm{e} = f_\mathrm{e}(x,t)$.

Vector Formulation

A Laplace transformation $\mathcal{L}$ is applied to the PDE (1), with $Y(x,s) = \mathcal{L}\{y(x,t)\}$. Subsequently the PDE is reformulated in a unifying vector form [5]

$$[s\boldsymbol{C} - L]\,\boldsymbol{Y}(x,s) = \boldsymbol{F}_\mathrm{e}(x,t), \qquad L = \boldsymbol{A} + \frac{\partial}{\partial x}\boldsymbol{I}, \tag{2}$$

with the spatial differential operator L, a mass matrix $\boldsymbol{C}$, a matrix of damping parameters $\boldsymbol{A}$ and the identity matrix $\boldsymbol{I}$. The vector of variables $\boldsymbol{Y}$ contains the independent variables of the system

$$\boldsymbol{Y}(x,s) = \begin{bmatrix} sY & Y'' & Y' & Y''' \end{bmatrix}^\mathrm{T}. \tag{3}$$

The matrices in (2) are of size 4×4 and result in

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ a_1 & a_2 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c_1 & c_2 & 0 & 0 \end{bmatrix}, \tag{4}$$

with the coefficients

$$a_1 = \frac{d_1}{EI} \quad a_2 = -\frac{T_\mathrm{s}}{EI} \quad c_1 = -\frac{\rho A}{EI} \quad c_2 = \frac{d_3}{EI}. \tag{5}$$

The equivalents between the scalar (1) and the vector form can be validated, by reinserting of (3) - (5) into (2).

Initial and Boundary Conditions

In addition to the PDE (1) an arbitrary set of boundary conditions is proposed for the design of a general string model in the continuous frequency domain

$$\begin{bmatrix} sY(x,s) \\ Y''(x,s) \end{bmatrix} = \begin{bmatrix} \Phi_1(x,s) \\ \Phi_2(x,s) \end{bmatrix} = \boldsymbol{\Phi}(x,s), \qquad x = 0, \ell. \tag{6}$$

For the string deflection y (and subsequently its time derivative $\dot{y}$) and the curvature y'' yet unspecified boundary values ϕ_1, ϕ_2 are assumed. To fit into the vector formulation (2) the boundary values are grouped into the vector $\boldsymbol{\Phi}$. These boundary conditions are used to design a general transfer function model of the vibrating string. Realistic boundary conditions are incorporated later by the modification of these boundary terms. Additionally it is assumed that the string deflection y and its derivatives are zero at $t = 0$.

All variables in y, which are not associated with the boundary conditions in (6), can be observed at $x = 0, \ell$. Following [6], the variables sY, Y'' can be interpreted as inputs of the system and Y', Y'' as outputs.

Sturm-Liouville Transformation

For the transformation of space variables, a Sturm-Liouville transformation is defined in terms of an integral transform [4]

$$\mathcal{T}\left\{\boldsymbol{Y}(x,s)\right\} = \bar{Y}(\mu,s) = \int_0^\ell \tilde{\boldsymbol{K}}^\mathrm{H}(x,\mu)\boldsymbol{C}\boldsymbol{Y}(x,s)\,\mathrm{d}x, \tag{7}$$

with the eigenfunctions $\tilde{\boldsymbol{K}}$. The inverse transformation is a generalized Fourier series with the eigenfunctions $\boldsymbol{K}$ and the scaling factor N_μ

$$\mathcal{T}^{-1}\left\{\bar{Y}(\mu,s)\right\} = \boldsymbol{Y}(x,s) = \sum_{\mu=0}^{\infty} \frac{1}{N_\mu}\bar{Y}(\mu,s)\boldsymbol{K}(x,\mu), \tag{8}$$

where $\bar{Y}(\mu,s)$ is a scalar representation of the vector of variables (2) in the temporal and spatial transform domain. Index $\mu \in \mathbb{Z}$ is the index of a spatial frequency variable s_μ for which (2) has nontrivial solutions [4, 7].

Applying the forward transformation (7) to the vector PDE (2) leads to a scalar representation of (2) in the spatio-temporal transform domain

$$s\bar{Y}(\mu,s) = s_\mu \bar{Y}(\mu,s) + \bar{\Phi}(\mu,s) + \bar{F}_\mathrm{e}(\mu,s), \tag{9}$$

where $\bar{\Phi}(\mu,s)$ is the transform domain representation of the boundary vector $\boldsymbol{\Phi}$ in (6) and $\bar{F}_\mathrm{e}(\mu,s)$ is the transform domain representation of the excitation functions. Eq. (9) already shows a strong similarity to the state equation of a standard state-space description. Grouping the variables in (9) into matrices and vectors leads to a state equation in vector form

$$s\bar{\boldsymbol{Y}}(s) = \boldsymbol{\mathcal{A}}\bar{\boldsymbol{Y}}(s) + \bar{\boldsymbol{F}}_\mathrm{e}(s) + \bar{\boldsymbol{\Phi}}(s), \tag{10}$$

where the dimensions of the vectors and matrices correspond to the range of the truncated sum in (8). The matrices and vectors in (10) are

$$\bar{\boldsymbol{Y}}(s) = \left[\ldots, \bar{Y}(\mu,s), \ldots\right]^\mathrm{T}, \quad \bar{\boldsymbol{\Phi}}(s) = \left[\ldots, \bar{\Phi}(\mu,s), \ldots\right]^\mathrm{T}, \tag{11}$$

$$\bar{\boldsymbol{F}}_\mathrm{e}(s) = \left[\ldots, \bar{F}_\mathrm{e}(\mu,s), \ldots\right]^\mathrm{T}, \quad \boldsymbol{\mathcal{A}} = \mathrm{diag}\left(\ldots, s_\mu, \ldots\right). \tag{12}$$

The matrix $\boldsymbol{\mathcal{A}}$ contains the eigenfrequencies of the system on its main diagonal. Reformulating the inverse transformation (8) into a vector form leads to the output equation of the state-space description

$$\boldsymbol{Y}(x,s) = \boldsymbol{\mathcal{C}}(x)\bar{\boldsymbol{Y}}(s), \quad \boldsymbol{\mathcal{C}}(x) = \left[\ldots, \tfrac{1}{N_\mu}\boldsymbol{K}(\mu,x), \ldots\right]. \tag{13}$$

The matrix $\boldsymbol{\mathcal{C}}(x)$ can be interpreted as a transformation matrix, transforming the vector $\bar{Y}$ in the spatio-temporal frequency domain back into the space domain.

With the state equation (10) and the output equation (13) a state-space description for the simulation of the vibrating string is realized (see Fig. 3).

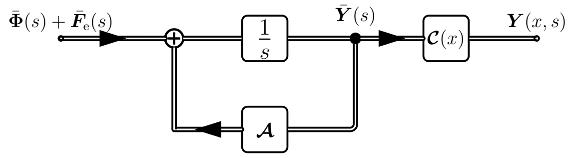

Figure 3: State-space representation of the string model according to (10), (13) with a general boundary term.

Realistic boundary conditions

The boundary behavior of the string is very complex. A guitar string is not just fixed at both ends, it is interacting with other oscillating systems. At the position $x = 0$, the string is attached to a bridge, which is connected to the body of the string. Also the hand plays a role for special playing styles, e.g. palm muted. At $x = \ell$, the string lies on the saddle and is wrapped around the mechanics at the guitar head (see Fig. 4). Additionally, for $0 < x < \ell$ the string interacts with the fingers and the fretboard.

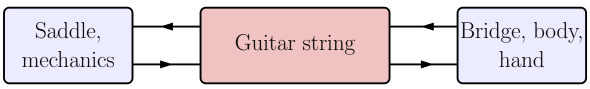

Figure 4: Interconnection of a guitar string to other sound-influencing parts of a guitar.

In this contribution, we focus on the incorporation of more realistic bridge/hand models at the string position $x = 0$. Therefore, two sets of boundary conditions are defined:

1. The string is fixed at both ends, which leads to a zero deflection and curvature at $x = 0, \ell$, which reads in the frequency domain

$$sY(x, s) = 0, \quad Y''(x, s) = 0, \quad x = 0, \ell. \tag{14}$$

2. The string is influenced at $x = 0$ by a frequency independent admittance Y_b

$$sY(0, s) - Y_\mathrm{b}\left[T_\mathrm{s}Y'(0, s) - EIY'''(0, s)\right] = 0, \tag{15}$$

   while the condition for $x = \ell$ is the same as in (14). This influence by a bridge impedance can serve as a very simple bridge model or as a model for the damping by the ball of the hand.

The first set of conditions (14) can also be realized by (15) with a zero impedance.

Incorporation of Boundary Conditions

The state-space description in (10), (13) describes the transversal wave propagation of a guitar string. But the exact boundary behavior was neglected. The model is designed with a generalized boundary term Φ, which is modified in this section to incorporate two different boundary conditions (14), (15) by techniques from control theory [7, 6, 8].

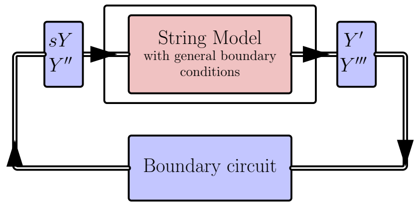

Figure 5: General concept for the incorporation of realistic boundary conditions. A general string model (red) is extended with a boundary circuit (blue) by the input and output variables.

The boundary conditions for fixed ends (14) can be easily realized by setting the boundary term $\bar{\Phi}$ in (10) to zero. The model with fixed ends serves as a ground truth when the influence of the impedance is investigated.
The incorporation of realistic boundary conditions (15) is performed by the design of a feedback loop (see Fig. 5) connecting the inputs (variables associated with bc) and outputs (observable variables) of the system [7]. The principle of the incorporation of (15) by the connection of inputs and outputs is shown in Fig. 6. The exact derivations are skipped here for brevity and can be found in [6, 9]. In the end, the influence of the boundary condi-

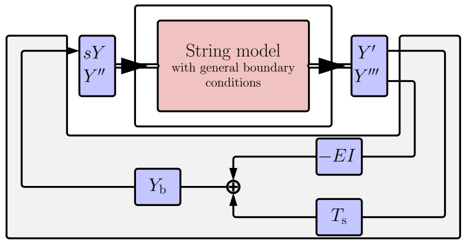

Figure 6: String model with impedance boundary conditions. General string model (red) is extended by a boundary circuit (grey) realizing the boundary conditions in (15)

tions is concentrated into a feedback matrix $\mathcal{BK}$, which modifies the state matrix $\mathcal{A}$ in the state equation (10) to

$$s\bar{Y}(s) = \mathcal{A}_\mathrm{c}\bar{Y}(s) + \bar{F}_\mathrm{e}(s), \quad \mathcal{A}_\mathrm{c} = (\mathcal{A} - \mathcal{BK}), \tag{16}$$

with a modified state matrix $\mathcal{A}_\mathrm{c}$. The output equation stays the same as (13). The structure of the modified state equation (16) is strongly related to control theory. The feedback matrix $\mathcal{BK}$ shifts the eigenvalues s_μ of the system to a certain position, so that the system fulfills the desired boundary behavior (see (15)).
Fig. 7 shows the eigenvalues s_μ in the complex plane for the boundary conditions in (14) (blue) and (15) (yellow). The figure shows the influence of the feedback matrix on the eigenvalues of the system. As the boundary conditions in (15) apply damping to each harmonic of the string, all poles are shifted in negative x-direction.

Experimental Verification

This section shows simulation results for different boundary conditions. The results are presented in terms of the amplitude spectrum of the bending force at the bride po-

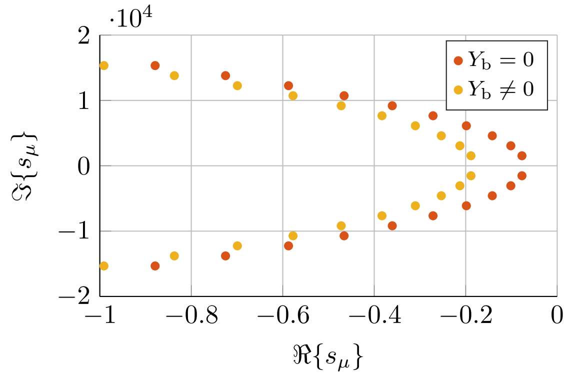

Figure 7: Eigenfrequencies s_μ of the string with a zero (red) and a non-zero (yellow, $Y_\mathrm{b} = 1 \cdot 10^{-2}$s/kg) bridge admittance Y_b in the complex plain.

sition $x = 0$

$$F_\mathrm{b}(0, s) = EIY'''(0, s), \tag{17}$$

where several valid simplifications were applied [2]. The simulations are preformed in the continuous frequency domain with the modified state equation (16) and the output equation (13). The parameters of the string for the matrices in (4) are shown in Table 1.
The function $f_\mathrm{e}(x, t)$ exciting the string is a single impulse at the position $x_\mathrm{e} = 0.3\,\mathrm{m}$ on the string, as shown in [9]. In the frequency domain this excitation leads to a uniform excitation of all frequency components. The simulation of output variables in response to this excitation can be seen as an impedance-analysis of the string.

ρ	Density	$7850\,\mathrm{kg/m^3}$
E	Young's modulus	$220\,\mathrm{GPa}$
l	Length	$0.648\,\mathrm{m}$
A	Cross section area	$0.0924\,\mathrm{mm^2}$
I	Moment of inertia	$6.7943 \cdot 10^{-4}\,\mathrm{mm^4}$
d_1	Freq. indep. damping	$10 \cdot 10^{-5}\,\mathrm{kg/(ms)}$
d_3	Freq. dep. damping	$0.5 \cdot 10^{-6}\,\mathrm{kgm/s}$
T_S	Tension	$72.2016\,\mathrm{N}$

Table 1: Physical parameters of an e-guitar steel B-string [10], with a fundamental frequency of $f_0 \approx 250$Hz.

Fig. 8 shows the amplitude spectra of the bending force for three different values of the admittance Y_b. Setting the admittance to zero in (15), one ends up directly by the boundary conditions for fixed ends in (14) (red curve). With this zero admittance, the fundamental and the harmonics of the string can be clearly detected.
Setting the admittance to a high value (blue curve in Fig 8) all harmonics are nearly completely damped. This effect may be also compared to the termination of an electrical transmission line by the characteristic impedance. Therefore, the sound of the string is nearly completely damped and the sound is dominated by the excitation, which leads to the sound of a palm-muted playing style. Choosing a mid-ranged admittance Y_b (yellow curve in Fig. 8), the same effect as in Fig. 7 can be observed. Each harmonic of the string is damped according to the admittance Y_b.

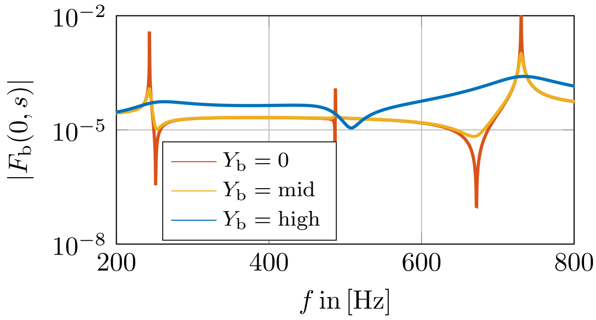

Figure 8: Amplitude spectra of the bending force F_b at bridge position $x = 0$ for different values of the bridge admittance $Y_\mathrm{b} = 0$ (red), $1 \cdot 10^{-2}$ s/kg (yellow), 2 s/kg (blue).

Conclusions and Further Works

This paper presented a model for a vibrating e-guitar string with realistic boundary conditions. The model is based on a modal expansion of a spatial differential operator and is represented in terms of a state-space description. Realistic boundary conditions are incorporated by the design of a feedback loop.
In further works the presented concepts are used to extend the model to more complex bridge and body models. Also the finger-string and string-fretboard interaction should be investigated.

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