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A Maximum-Achievable-Directivity Beamformer with White-Noise-Gain Constraint for Spherical Microphone Arrays

Xi Chen(1), Gongping Huang(1), Jingdong Chen(1), and Jacob Benesty(2)

(1)Center of Intelligent Acoustics and Immersive Communications, Northwestern Polytechnical University, Xi'an, 710072, China. E-mail: xichen128@mail.nwpu.edu.cn, gongpinghuang@gmail.com, jingdongchen@ieee.org

(2)INRS-EMT, University of Quebec, 800 de la Gauchetiere Ouest, Suite 6900, Montreal, QC H5A 1K6, Canada. E-mail: benesty@emt.inrs.ca

Abstract

In microphone array beamforming, it is desirable to achieve a directive gain as high as possible for maximum acoustic noise rejection. The well-known superdirective beamformer was developed for this purpose; but it is sensitive to array imperfections such as sensors' self noise, sensor placement errors, mismatch among sensor responses, etc, which restrict its application in practical systems. To circumvent the robustness issue, we recently developed a flexible high directivity beamforming method that can achieve a flexible compromise between the directivity factor (DF) and the level of white noise gain (WNG) by adjusting the value of a control parameter. This principle is further extended in this work. We present a beamforming method with spherical microphone arrays. It first determines the maximum order (consequently the largest value of DF) that can be achieved in an analytical way based on a specified level of WNG that is tolerable by the array in the frequency band of interest. This order is then used to determine the value of a control parameter. The maximum-achievable-directivity beamformer is then designed. The advantage of this beamformer over the existing ones is that it is robust to implement and yet able to achieve the maximum possible DF.

Keywords: Spherical microphone arrays, directivity factor, superdirective beamformer, white noise gain.

1 INTRODUCTION

Microphone array beamforming, which aims at enhancing acoustic and speech signals from a desired look direction and attenuating noise and interference from other directions, is an important topic in speech processing [1–4]. Many algorithms have been developed including but not limited to the delay-and-sum beamformer, the superdirective beamformer [5–7], differential beamformers [8–13], and the modal beamformers [14, 15]. Generally, the beamformer's performance depends not only on the beamforming algorithm used, but also on the array geometry. In many practical applications such as teleconferencing and three-dimensional sound recording and reproduction, spherical microphone arrays are generally preferred because they can have similar responses across different directions in the three-dimensional space. Among many beamforming methods developed for spherical microphone arrays [2, 15–21], the superdirective beamformer, which maximizes the directivity factor (DF), is effective in suppressing directional noise and reverberation and therefore has great potential to be used in noisy auditory scenes [2, 22–24]. However, the lack of robustness is a serious issue that prevents this beamformer from being widely used in a wide range of acoustic applications. In our previous work, we proposed a flexible high directivity beamformer, which attempts to a compromise between the DF and the white noise gain (WNG) [25]. In this paper, we extend the idea to design a maximum-achievable-directivity beamformer with spherical microphone arrays that can achieve the largest possible DF and maintain the WNG larger than a pre-specified level.

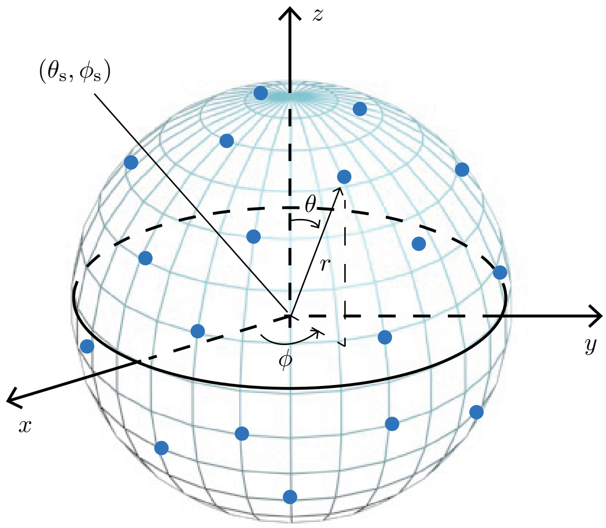


Figure 1. Illustration of a spherical microphone array in the Cartesian coordinate system, where (θ_s, ϕ_s) denotes the source incidence direction.

2 SIGNAL MODEL AND PROBLEM FORMULATION

We consider a spherical microphone array with a radius of r, which consists of M omnidirectional microphones. We assume that the center of the spherical microphone array coincides with the origin of the Cartesian coordinate system as shown in Fig.1. The azimuth angle ϕ is measured counterclockwise from the x-axis, while the elevation angle θ is measured downwards from the z-axis. Suppose that we want to steer the beamformer to the direction (θ, ϕ), the steering vector of length M is written as

$$\mathbf{d}(\omega, \theta, \phi) = \begin{bmatrix} e^{\jmath\varpi\mathbf{p}^T\mathbf{r}_1} & e^{\jmath\varpi\mathbf{p}^T\mathbf{r}_2} & \cdots & e^{\jmath\varpi\mathbf{p}^T\mathbf{r}_M} \end{bmatrix}^T, \tag{1}$$

where the superscript ^{T} is the transpose operator, $\jmath$ is the imaginary unit with $\jmath^2 = -1$, $\varpi = \omega r/c$, $\omega = 2\pi f$ is the angular frequency, $f > 0$ is the temporal frequency, c is the speed of sound in the air ($c = 340$ m/s), $\mathbf{p} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{bmatrix}^T$, and $\mathbf{r}_m = \begin{bmatrix} \sin\theta_m\cos\phi_m & \sin\theta_m\sin\phi_m & \cos\theta_m \end{bmatrix}^T$.

Assume that the source signal of interest propagates from the direction (θ_s, ϕ_s). The observation signal vector can be written as

$$\begin{aligned} \mathbf{y}(\omega) &= \begin{bmatrix} Y_1(\omega) & Y_2(\omega) & \cdots & Y_M(\omega) \end{bmatrix}^T \\ &= \mathbf{d}(\omega, \theta_s, \phi_s) X(\omega) + \mathbf{v}(\omega), \end{aligned} \tag{2}$$

where $Y_m(\omega)$, $m = 1, 2, \ldots, M$ is the signal observed by the mth microphone, $X(\omega)$ is the source signal of interest (also called the desired signal), and $\mathbf{v}(\omega)$ is the additive noise vector.

Beamforming is achieved by applying a complex-valued weight to each microphone signal and then summing up the weighted outputs to obtain an estimate of the desired source signal, i.e.,

$$\begin{aligned} Z(\omega) &= \sum_{m=1}^{M} H_m^*(\omega) Y_m(\omega) \\ &= \mathbf{h}^H(\omega)\mathbf{y}(\omega) \\ &= \mathbf{h}^H(\omega)\mathbf{d}(\omega, \theta_s, \phi_s) X(\omega) + \mathbf{h}^H(\omega)\mathbf{v}(\omega), \end{aligned} \tag{3}$$

where the superscript * is the complex-conjugate operator, the superscript ^{H} is the conjugate-transpose operator, and

$$\mathbf{h}(\omega) = \begin{bmatrix} H_1(\omega) & H_2(\omega) & \cdots & H_M(\omega) \end{bmatrix}^T \tag{4}$$

is the beamforming filter of length M.

The objective of beamforming is to design $\mathbf{h}(\omega)$ such that $Z(\omega)$ is a good estimate of $X(\omega)$. Generally, the distortionless constraint at the desired look direction, i.e.,

$$\mathbf{h}^H(\omega)\,\mathbf{d}(\omega, \theta_\mathrm{s}, \phi_\mathrm{s}) = 1, \tag{5}$$

should be used so that the desired signal can pass through the beamformer without being distorted while noise and interference from directions other than the desired look direction are attenuated.

3 MAXIMUM-ACHIEVABLE-DIRECTIVITY BEAMFORMER WITH WHITE-NOISE-GAIN CONSTRAINT

The beampattern, which depicts the response of the array to the signal incident from the direction (θ, ϕ), is defined as

$$\begin{aligned} \mathcal{B}[\mathbf{h}(\omega), \theta, \phi] &= \mathbf{h}^H(\omega)\,\mathbf{d}(\omega, \theta, \phi) \\ &= \sum_{m=1}^{M} H_m^*(\omega)\, e^{\jmath\varpi \mathbf{p}^T \mathbf{r}_m}. \end{aligned} \tag{6}$$

The ideal beampattern is expected to have a unit impulse at the desired look direction and be zero elsewhere, i.e.,

$$\mathcal{B}(\theta, \phi) = \delta(\cos\theta - \cos\theta_s)\,\delta(\phi - \phi_s), \tag{7}$$

where $\delta(\cdot)$ denotes the Dirac delta function. In practice, this beampattern is impossible to achieve, so we approximate it by a series of spherical harmonics up to the Nth order [19], i.e.,

$$\mathcal{B}(\theta, \phi) \approx \mathcal{B}_N(\theta, \phi) = \sum_{n=0}^{N}\sum_{l=-n}^{n} \left[Y_n^l(\theta_\mathrm{s}, \phi_\mathrm{s})\right]^* Y_n^l(\theta, \phi), \tag{8}$$

where

$$Y_n^l(\theta, \phi) = \sqrt{\frac{2n+1}{4\pi}\frac{(n-l)!}{(n+l)!}}\,\mathcal{P}_n^l(\cos\theta)\, e^{\jmath l\phi} \tag{9}$$

denotes the spherical harmonics of order $n\,(= 0, 1, 2, \ldots, +\infty)$ and degree $l\,(= 0, \pm1, \ldots, \pm n)$, with $(\cdot)!$ being the factorial function and $\mathcal{P}_n^l(\cos\theta)$ being the associated Legendre function of order n and degree l [26].

One way to design beamformers is to optimize the filter, $\mathbf{h}(\omega)$, to make the designed beampattern, i.e., $\mathcal{B}[\mathbf{h}(\omega), \theta, \phi]$, as close as possible to $\mathcal{B}_N(\theta, \phi)$. However, the resulting beamformer generally suffers from serious white noise amplification at low frequencies. To achieve a flexible compromise between the DF and the level of WNG, a flexible high directivity beampattern is defined in [25]

$$\begin{aligned} \mathcal{B}_{N,\alpha}(\theta, \phi) &= \frac{1}{\kappa_N}\left[\alpha\mathcal{B}_N(\theta, \phi) + (1-\alpha)\mathcal{B}_{N-1}(\theta, \phi)\right] \\ &= \frac{\alpha}{\kappa_N}\sum_{n=0}^{N}\sum_{l=-n}^{n}\left[Y_n^l(\theta_\mathrm{s}, \phi_\mathrm{s})\right]^* Y_n^l(\theta, \phi) + \frac{1-\alpha}{\kappa_N}\sum_{n=0}^{N-1}\sum_{l=-n}^{n}\left[Y_n^l(\theta_\mathrm{s}, \phi_\mathrm{s})\right]^* Y_n^l(\theta, \phi) \\ &= \frac{1}{\kappa_N}\sum_{n=0}^{N}\sum_{l=-n}^{n}\xi_n\left[Y_n^l(\theta_\mathrm{s}, \phi_\mathrm{s})\right]^* Y_n^l(\theta, \phi), \end{aligned} \tag{10}$$

where $\alpha \in [0,1]$ is a real-valued control parameter, $\kappa_N = \frac{1}{4\pi}\left[N^2 + \alpha(2N+1)\right]$ is the corresponding normalized factor, $\xi_n = 1$, for $n = 0, 1, \ldots, N-1$, and $\xi_n = \alpha$, for $n = N$. Clearly, if $\alpha = 0$, $\mathcal{B}_{N,\alpha} = \mathcal{B}_{N-1}(\theta, \phi)$, and if $\alpha = 1$, $\mathcal{B}_{N,\alpha} = \mathcal{B}_N(\theta, \phi)$. As the value of α increases from 0 to 1, the DF of the resulting beamformer increases; but the noise amplification problem may become more serious.

The WNG, which evaluates the robustness of the beamformer to the array's imperfection [2,3], is defined as

$$\mathcal{W}[\mathbf{h}(\omega)] = \frac{\left|\mathbf{h}^H(\omega)\mathbf{d}(\omega, \theta_s, \phi_s)\right|^2}{\mathbf{h}^H(\omega)\mathbf{h}(\omega)}. \tag{11}$$

In this paper, we propose a maximum-achievable-directivity (MAD) beamformer, which can obtain the maximum possible directivity while maintains a specified level of WNG for robustness. This problem can be described as

$$\mathbf{h}_{N,\alpha}(\omega) = \underset{\mathbf{h}(\omega)}{\arg\max}\, \mathcal{W}[\mathbf{h}(\omega)] \quad \text{s.t.} \quad \begin{cases} \mathcal{B}[\mathbf{h}_{N,\alpha}(\omega), \theta, \phi] = \mathcal{B}_{N,\alpha}(\theta, \phi) \\ \mathcal{W}[\mathbf{h}_{N,\alpha}(\omega)] \geq \mathcal{W}_0 \end{cases}, \tag{12}$$

which can be equivalently expressed [assuming that the distortionless constraint is fulfilled in $\mathcal{B}_{N,\alpha}(\theta, \phi)$] as

$$\mathbf{h}_{N,\alpha}(\omega) = \underset{\mathbf{h}(\omega)}{\arg\min}\, \mathbf{h}^H(\omega)\mathbf{h}(\omega) \quad \text{s.t.} \quad \begin{cases} \mathcal{B}[\mathbf{h}_{N,\alpha}(\omega), \theta, \phi] = \mathcal{B}_{N,\alpha}(\theta, \phi) \\ \mathcal{W}[\mathbf{h}_{N,\alpha}(\omega)] \geq \mathcal{W}_0 \end{cases}, \tag{13}$$

where $\mathcal{W}_0$ denotes the minimum level of WNG.

Using the spherical harmonic expansion, we have

$$e^{\jmath\varpi\mathbf{p}^T\mathbf{r}_m} = \sum_{n=0}^{\infty}\sum_{l=-n}^{n} 4\pi\jmath^n \mathcal{J}_n(\varpi) \times \left[Y_n^l(\theta_m, \phi_m)\right]^* Y_n^l(\theta, \phi), \tag{14}$$

where $\mathcal{J}_n(\varpi)$ is the nth-order spherical bessel functions of the first kind. Substituting (14) into (6), we deduce that

$$\begin{aligned} \mathcal{B}[\mathbf{h}(\omega), \theta, \phi] &= \sum_{m=1}^{M} H_m^*(\omega) e^{\jmath\varpi\mathbf{p}^T\mathbf{r}_m} \\ &\approx \sum_{m=1}^{M} H_m^*(\omega) \sum_{n=0}^{N}\sum_{l=-n}^{n} 4\pi\jmath^n \mathcal{J}_n(\varpi) \times \left[Y_n^l(\theta_m, \phi_m)\right]^* Y_n^l(\theta, \phi) \\ &= \sum_{n=0}^{N}\sum_{l=-n}^{n} Y_n^l(\theta, \phi)\, 4\pi\jmath^n \mathcal{J}_n(\varpi) \times \sum_{m=1}^{M} H_m^*(\omega)\left[Y_n^l(\theta_m, \phi_m)\right]^*. \end{aligned} \tag{15}$$

Comparing the result in (15) and the target beampattern $\mathcal{B}_{N,\alpha}(\theta, \phi)$ in (10), we get the following relationship:

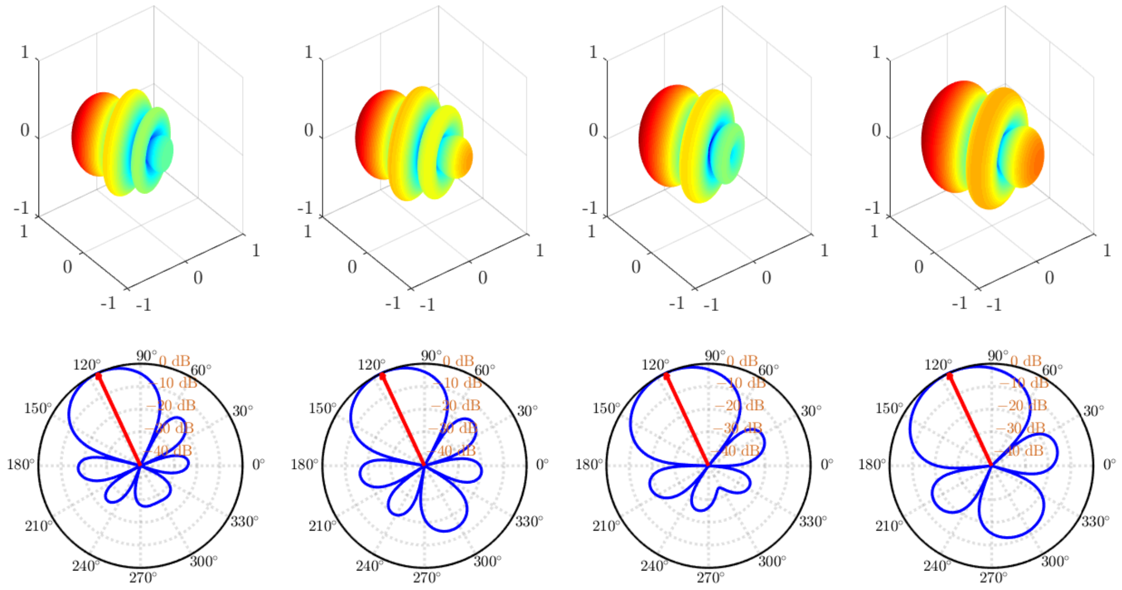
$$4\pi\left[\jmath^n \mathcal{J}_n(\varpi)\right]^* \sum_{m=1}^{M} H_m(\omega) Y_n^l(\theta_m, \phi_m) = \frac{\xi_n}{\kappa_N} Y_n^l(\theta_s, \phi_s). \tag{16}$$

It follows immediately that

$$\mathbf{Y}\mathbf{h}(\omega) = \boldsymbol{\eta}(\varpi), \tag{17}$$

where

$$\mathbf{Y} = \begin{bmatrix} Y_0^0(\theta_1, \phi_1) & Y_0^0(\theta_2, \phi_2) & \cdots & Y_0^0(\theta_M, \phi_M) \\ Y_1^{-1}(\theta_1, \phi_1) & Y_1^{-1}(\theta_2, \phi_2) & \cdots & Y_1^{-1}(\theta_M, \phi_M) \\ Y_1^0(\theta_1, \phi_1) & Y_1^0(\theta_2, \phi_2) & \cdots & Y_1^0(\theta_M, \phi_M) \\ Y_1^1(\theta_1, \phi_1) & Y_1^1(\theta_2, \phi_2) & \cdots & Y_1^1(\theta_M, \phi_M) \\ \vdots & \vdots & \ddots & \vdots \\ Y_N^N(\theta_1, \phi_1) & Y_N^N(\theta_2, \phi_2) & \cdots & Y_N^N(\theta_M, \phi_M) \end{bmatrix} \tag{18}$$



(a) (b) (c) (d)

Figure 2. Beampatterns of the MAD beamformer with different values of $\mathcal{W}_0$: (a) $\mathcal{W}_0 = -40$ dB, (b) $\mathcal{W}_0 = -30$ dB, (c) $\mathcal{W}_0 = -20$ dB, and (d) $\mathcal{W}_0 = -10$ dB. Conditions of simulations: $M = 32$, $r = 3$ cm, $f = 1$ kHz, and $(\theta_{\rm s}, \phi_{\rm s}) = (90°, 115°)$.

is a matrix of size $(N+1)^2 \times M$, and

$$\boldsymbol{\eta}(\varpi) = \frac{1}{\kappa_N} \begin{bmatrix} \frac{\xi_0 Y_0^0(\theta_{\rm s},\phi_{\rm s})}{4\pi \mathcal{J}_0^*(\varpi)} & \frac{\xi_1 Y_1^{-1}(\theta_{\rm s},\phi_{\rm s})}{4\pi [\jmath \mathcal{J}_1(\varpi)]^*} & \cdots & \frac{\xi_N Y_N^N(\theta_{\rm s},\phi_{\rm s})}{4\pi [\jmath^N \mathcal{J}_N(\varpi)]^*} \end{bmatrix}^T \tag{19}$$

is a vector of length $(N+1)^2$.

Generally, $M \geq (N+1)^2$, the minimum-norm solution of (17) is

$$\mathbf{h}_{N,\alpha}(\omega) = \mathbf{Y}^H \left(\mathbf{Y}\mathbf{Y}^H\right)^{-1} \boldsymbol{\eta}(\varpi), \tag{20}$$

which gives the Nth-order MAD beamformer.

The corresponding WNG of the Nth-order MAD beamformer is

$$\mathcal{W}[\mathbf{h}_{N,\alpha}(\omega)] = \frac{M\left[N^2 + (2N+1)\,\alpha\right]^2}{\frac{(2N+1)}{\mathcal{J}_N^2(\varpi)}\alpha^2 + \sum_{n=0}^{N-1} \frac{(2n+1)}{\mathcal{J}_n^2(\varpi)}}. \tag{21}$$

Clearly, the WNG of the beamformer is a decreasing function of α and satisfies

$$\frac{M(N+1)^4}{\sum_{n=0}^{N} \frac{(2n+1)}{\mathcal{J}_n^2(\varpi)}} \leq \mathcal{W}[\mathbf{h}_{N,\alpha}(\omega)] \leq \frac{MN^4}{\sum_{n=0}^{N-1} \frac{(2n+1)}{\mathcal{J}_n^2(\varpi)}}. \tag{22}$$

In practice, we should first determine the order N and the value of the parameter α according to $\mathcal{W}_0$. Assume that the maximum designable order we want to achieve is $N_{\rm M}$, the order N and the parameter α can be determined as follows.

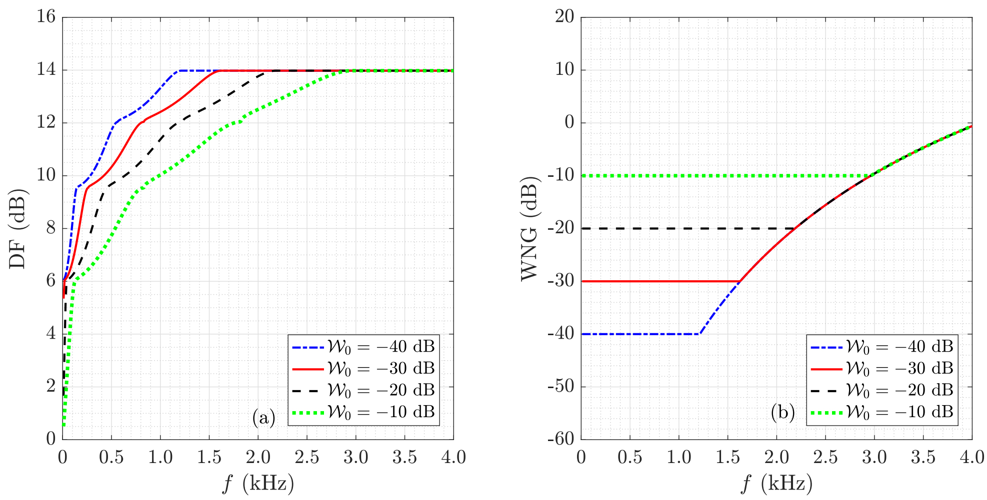


Figure 3. DFs and WNGs of the MAD beamformer as a function of the frequency, f: (a) DF and (b) WNG. Conditions of simulation: $M = 32$ and $r = 3$ cm.

- If $\mathcal{W}_0 < M(N_\mathrm{M}+1)^4/(\sum_{n=0}^{N_\mathrm{M}} \frac{(2n+1)}{\mathcal{J}_n^2(\varpi)})$, we simply choose $N = N_\mathrm{M}$ and $\alpha = 1$.
- If $\mathcal{W}_0 \in \left[M(N_\mathrm{M}+1)^4/(\sum_{n=0}^{N_\mathrm{M}} \frac{(2n+1)}{\mathcal{J}_n^2(\varpi)}), M\right]$, we first determine the order N as

$$N = \arg_{N'} \frac{M(N'+1)^4}{\sum_{n=0}^{N'} \frac{(2n+1)}{\mathcal{J}_n^2(\varpi)}} \leq \mathcal{W}_0 \leq \frac{MN'^4}{\sum_{n=0}^{N'-1} \frac{(2n+1)}{\mathcal{J}_n^2(\varpi)}}, \tag{23}$$

and then determine the value of the parameter α according to the following equation:

$$\frac{M\left[N^2 + (2N+1)\,\alpha\right]^2}{\frac{(2N+1)}{\mathcal{J}_N^2(\varpi)}\alpha^2 + \sum_{n=0}^{N-1}\frac{(2n+1)}{\mathcal{J}_n^2(\varpi)}} = \mathcal{W}_0, \tag{24}$$

whose solution is

$$\alpha = \frac{MN^4(2N+1) + \sqrt{\mathcal{W}_0 M N^4 \frac{(2N+1)}{\mathcal{J}_N^2(\varpi)} - \left[\mathcal{W}_0{}^2 \frac{(2N+1)}{\mathcal{J}_N^2(\varpi)} - \mathcal{W}_0 M (2N+1)^2\right] \sum_{n=0}^{N-1} \frac{(2n+1)}{\mathcal{J}_n^2(\varpi)}}}{\mathcal{W}_0 \frac{2N+1}{\mathcal{J}_N^2(\varpi)} - M(2N+1)^2}. \tag{25}$$

With the determined order N and the value of the parameter α, the MAD beamformer is subsequently obtained according to (20), which is robust to implement and can achieve the maximum possible DF with a specified level of WNG.

4 SIMULATIONS

In this section, we evaluate the performance of the MAD beamformer. We consider a spherical microphone array with a radius of $r = 3$ cm, which consists of 32 omnidirectional microphones. The maximum designable order of the MAD beamformer is $N_\mathrm{M} = 4$. The desired steering direction is $(90°, 115°)$.

Figure 2 plots the beampatterns of the MAD beamformer for $\mathcal{W}_0 \in \{-40\ \text{dB}, -30\ \text{dB},\ -20\ \text{dB}, -10\ \text{dB}\}$, $f = 1$ kHz, where the upper subplot shows the beampatterns in the three-dimensional space and the lower subplot shows the corresponding two-dimensional beampatterns relative to ϕ. It is seen that the beampattern varies with the value of $\mathcal{W}_0$ and a larger value of $\mathcal{W}_0$ corresponds to a narrower main beam.

Figure 3 plots the DFs and the WNGs of the MAD beamformer as a function of the frequency. It is clearly seen that the designed beamformer can make the WNG larger than the specified level over the frequency band of interest. It is also seen the DF is improved with the increase of the frequency, and the maximum DF is $10\log_{10}(N_{\mathrm{M}} + 1)^2 \approx 13.98$ dB, which corresponds to the fourth-order MAD beamformer.

5 CONCLUSIONS

In this paper, we developed a MAD beamformer with spherical microphone arrays. It first determines the order of the beamformer based on the minimum level of WNG that is tolerable by the beamformer in the frequency band of interest. This order is then used to determine the value of a control parameter with a specified level of WNG. Both the order and the control parameter are determined in an analytic form. The MAD beamformer is subsequently designed, which is able to achieve the maximum possible DF while keeping the WNG larger than the specified level over the frequency band of interest. This MAD beamformer is robust to implement in practice.

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