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Physical realization of the radiation of complex multipoles

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ABSTRACT

Complex multipoles are sound sources located at positions having an imaginary component. They are also solutions of the Helmholtz equation, but their radiation patterns are different from those of their real counterparts. If $\vec{r}_q^{\,c}$ is the complex source position, the source preferably emits the sound in a direction indicated by Im($\vec{r}_q^{\,c}$) and the degree of focusing depends on the amplitude of Im($\vec{r}_q^{\,c}$). A loudspeaker in a box behaves at low frequencies like a monopole, a loudspeaker without a box like a dipole and a tuning fork like a longitudinal quadrupole. In contrast, complex multipoles do not have simple physical equivalents, since they are pure mathematical constructions. In the present work, the sound radiation of complex multipoles will be modelled by an array of loudspeakers whose amplitudes are determined so, that their radiation pattern is reproduced. To obtain the amplitude of each loudspeaker, a combination of the Boundary Element Method (BEM) and the Equivalent Source Method (ESM) is applied. With the BEM the transfer function of each loudspeaker is calculated and with the ESM their amplitude by minimizing the error at certain control points.

Keywords: Radiation pattern, Complex position, BEM, ESM

1. INTRODUCTION

Complex multipoles can be of interest for practical applications, since they possess a pronounced directivity, which depends on two parameters, a scalar and a unit vector. But to be used, they need to be physically constructable. Norris (1) showed that a monopole can be represented by a distribution of monopoles on the surface of a sphere of radius a, with a uniform surface density $(4\pi a^2 j_0(ka))^{-1}$. By shifting the radius a into the complex plane, he demonstrated that the monopole could also be represented as a distribution of complex monopoles with imaginary part Im(a) over the surface of a sphere with radius Re(a) and unit vectors parallel to the radial unit vectors. Unfortunately, the opposite situation, i.e. a representation of a complex monopole in terms of a uniform distribution of real monopoles is not possible.

The next possible situation is the representation of a complex multipole with a non uniform surface distribution. In practice, this can be achieved by an array of loudspeakers flush-mounted on the surface of a radiating body. The radiation of each loudspeaker has to be controlled in such a way that the total emitted sound pressure reproduces the radiation of the complex monopole. In the present work, we develop a procedure to obtain the normal velocity of the loudspeaker membranes of the radiator, based on the “Dummy-source-method” conceived by Pavić and Lindberg (2). They replace the original radiating body by a rigid dummy source with a simpler geometry and occupying (approximately) the same volume, covered by several vibrating patches on the surface. The radiation of the patches is obtained using the equivalent source method (ESM). In our case, we are in principle not limited to any particular object size, since our original source is a mathematical construction and we use a combination of BEM and ESM to compute the normal velocity of the loudspeaker membranes.

2. COMPLEX SOURCE POSITIONS

For the time convention $e^{j\omega t}$, a complex source position $\vec{r}_q^{\,c}$ is defined as

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$$\vec{r}_q^{\,c} = \vec{r}_q - j\beta\hat{u} \qquad , \qquad \text{with} \quad j = \sqrt{-1} \tag{1}$$

where $\vec{r}_q$ is the real part and $\beta\hat{u}$ ($\hat{u}$: unit vector) is the imaginary part.

The distance between a (real) field point $\vec{r}$ and the complex source position $\vec{r}_q^{\,c}$ is defined by the distance function R^c

$$R^c = \sqrt{\left(\vec{r} - \vec{r}_q^{\,c}\right)\cdot\left(\vec{r} - \vec{r}_q^{\,c}\right)} \tag{2}$$

In this case, R^c vanishes not only in a point, but in a ring with radius β (singularity ring) that lies in the plane

$$\left(\vec{r} - \vec{r}_q\right)\cdot\hat{u} = 0 \tag{3}$$

In order to have outgoing waves, Re(R^c)>0. This condition is fulfilled by choosing the branch cut

$$-\frac{\pi}{2} < \arg\left(R^c\right) < \frac{\pi}{2} \tag{4}$$

As a result of the choice, arg(R^c) changes from +π/2 at points slightly above, to -π/2 at points slightly below a circular disk bounded by the singularity ring, i.e. a jump of Im(R^c) is obtained (Fig. 1).

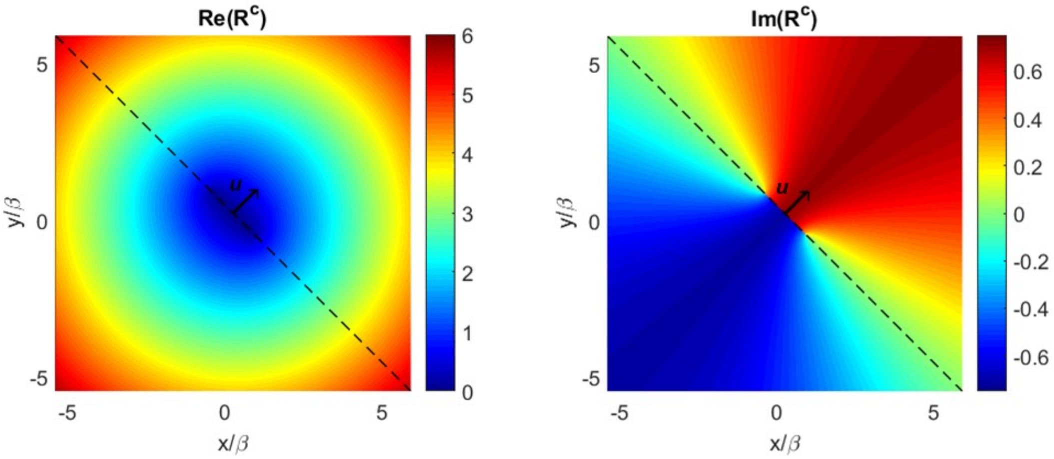

Figure 1 – Real and imaginary part of R^c with $\hat{u} = (1,1,0)/\sqrt{2}$

3. COMPLEX MULTIPOLES

Complex multipoles can be generated by moving the source position in the complex plane. Hence, one defines the complex radiator of zero-th order, the complex monopole, as

$$p_{000} = \frac{e^{-jkR^c}}{4\pi R^c} \tag{5}$$

With the choice of the branch cut (4), one recognizes that the complex monopole has a singularity ring and radiates with a higher amplitude on the positive side of the jump ($\mathrm{Im}(R^c) > 0$) than on the negative side ($\mathrm{Im}(R^c) < 0$). So, the complex monopole shows a pronounced sound radiation in the direction of the unit vector $\hat{u}$. The focusing of the radiated sound rises with increasing $k\beta$ as well as the size of the singularity ring. Fig. 2 shows the dependence of the focusing on β and on the frequency.

Complex multipoles can be generated from the complex monopole. After Shin and Felsen (3), the complex multipole of order (*m*, *n*, *l*) can be obtained through repeated differentiation of the monopole

(p_{000}) with respect to *x*, *y* and *z*.

$$p_{nml} = \frac{\partial^n}{\partial x^n}\frac{\partial^m}{\partial x^m}\frac{\partial^l}{\partial x^l}p_{000} \tag{6}$$

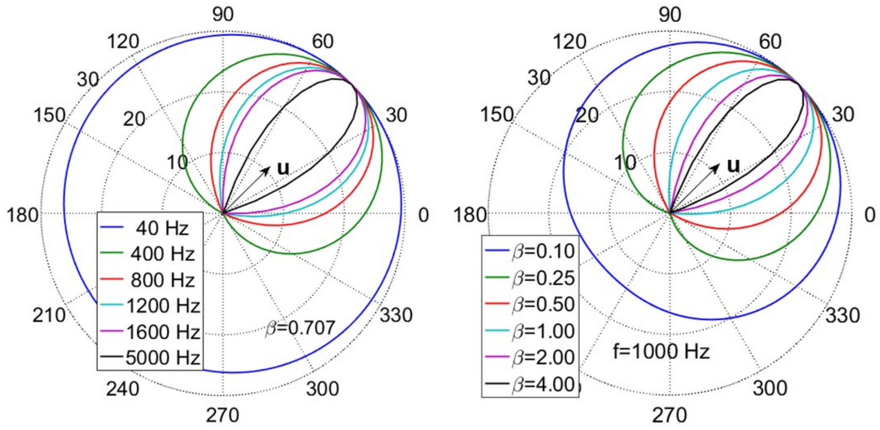

Figure 2 – Radiation pattern in the far field of a complex monopole - left: fixed β ; right: fixed frequency

As example, Fig. 3 shows the radiation patterns of the dipole p_{100} and the quadrupole p_{110} whose formulas are given in Eqs. (7) and (8)

$$p_{100} = -\left(jk + \frac{1}{R^c}\right)\frac{\left(x - x_q^c\right)}{R^c}p_{000} \tag{7}$$

$$p_{110} = \left(-k^2 + \frac{3jk}{R^c} + \frac{3}{\left(R^c\right)^2}\right)\frac{\left(x - x_q^c\right)}{R^c}\frac{\left(y - y_q^c\right)}{R^c}p_{000} \tag{8}$$

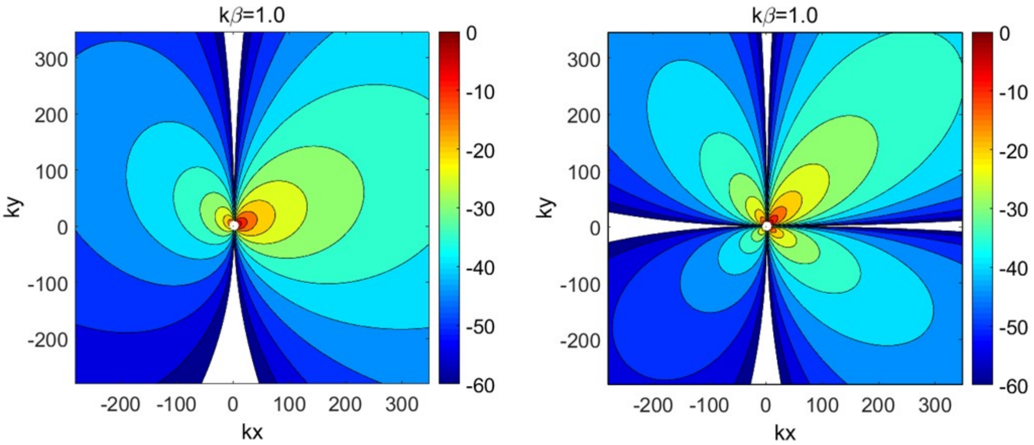

Figure 3 – Radiation pattern of complex multipoles - left: dipole p_{100} ; right: quadrupole p_{110}

4. REALIZATION OF THE RADIATION PATTERN

Complex multipoles are interesting, because they can focus the acoustic energy in a specified direction, even at low frequencies. Since they are mathematical constructions, there are no physical mechanisms that can produce such radiation patterns. However it is possible to reconstruct their sound fields with an array of loudspeakers. Existing arrays are mostly constructed to radiate the sound in a half space. In the present work, an array is developed that can focus the sound at any direction in space.

That is why we consider spherical polyhedra. Three models are investigated: a) dodecahedron; b) icosahedron and c) truncated icosahedron (“football”). The 3 models have a diameter of about 50 cm and are equipped with loudspeakers. The number of loudspeakers (*L*) and their diameter (D_L) are: *L*=12 and D_L =18.3 cm for the dodecahedron, L=20 and D_L =13.7 cm for the icosahedron and L=32 and D_L =11 cm for the truncated icosahedron (Fig. 4).

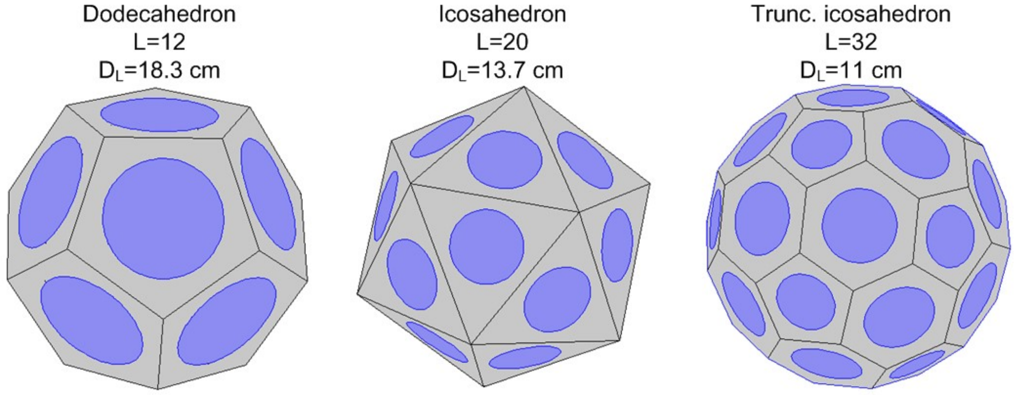

Figure 4 Different models of the loudspeaker array

It is supposed that each loudspeaker has following radial normal velocity distribution

$$v_n = \frac{3}{\pi (D_L/2)^2}\left(1-\left(\frac{r}{D_L/2}\right)^2\right)^2 \tag{9}$$

For the reconstruction of the radiation pattern of the complex multipoles, the amplitude and phase of each loudspeaker will be determined. The method is a variation of the „Dummy-source-method“ and consists of the following steps:

- Simulation of the sound pressure at C control points around the model (Fig. 5), produced by one loudspeaker when all others are turned off, using the BEM. A $C\times L$ matrix (***T***) is built with the values of sound pressure from all loudspeakers.

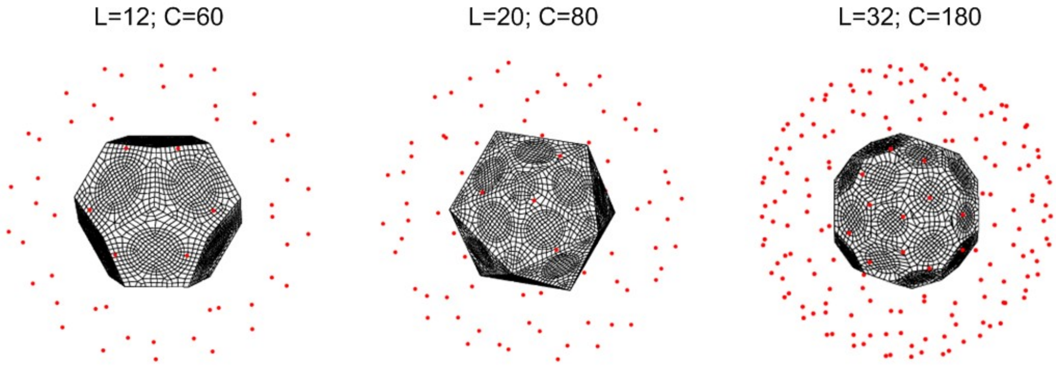

Figure 5 Control points for the three models

- Computation of the exact values of the sound pressure at the control points using the analytical formulas. These values build a vector ***p*** of length L
- Calculation of the amplitude and phase of each loudspeaker by solving an overdetermined system of equations (10). These values build the vector of unknowns ***q*** of length L.

$$Tq = p \tag{10}$$

To evaluate the quality of the simulation, we compute the averaged relative error percentage ε_C at the control points

$$\varepsilon_C = \sqrt{\frac{\sum_{i=1}^{D}\left|p_i^{(s)} - p_i^{(e)}\right|^2}{\sum_{i=1}^{D}\left|p_i^{(e)}\right|^2}} \times 100 \tag{11}$$

where $p_i^{(s)}$ and $p_i^{(e)}$ are the simulated and exact values of the sound pressure at the i-th control point respectively.

5. RESULTS

A simulation with the three models for a complex monopole with imaginary amplitude $\beta = 0.2$ was performed. This value of β ensures that the singularity ring is completely contained inside the models. The control points are located at surfaces that have the same form of the models but double the size. Hence, a good agreement between simulation and exact values is expected in the region outside the control points.

Table 1 shows the error percentage obtained with each of the described models.

Table 1 – Relative error percentage at the control points ε_C

Nr. loudspeakers	125 Hz	250 Hz	500 Hz	1000 Hz	2000 Hz
12	2.6	3.3	9.3	54.2	66.2
20	1.5	1.9	5.8	42.7	78.5
32	0.3	0.3	0.5	4.9	72.7

The errors below 10% are presented in blue and the errors above 10% in red. The errors drop with increasing number of loudspeakers when the models have at least two loudspeakers per wavelength.

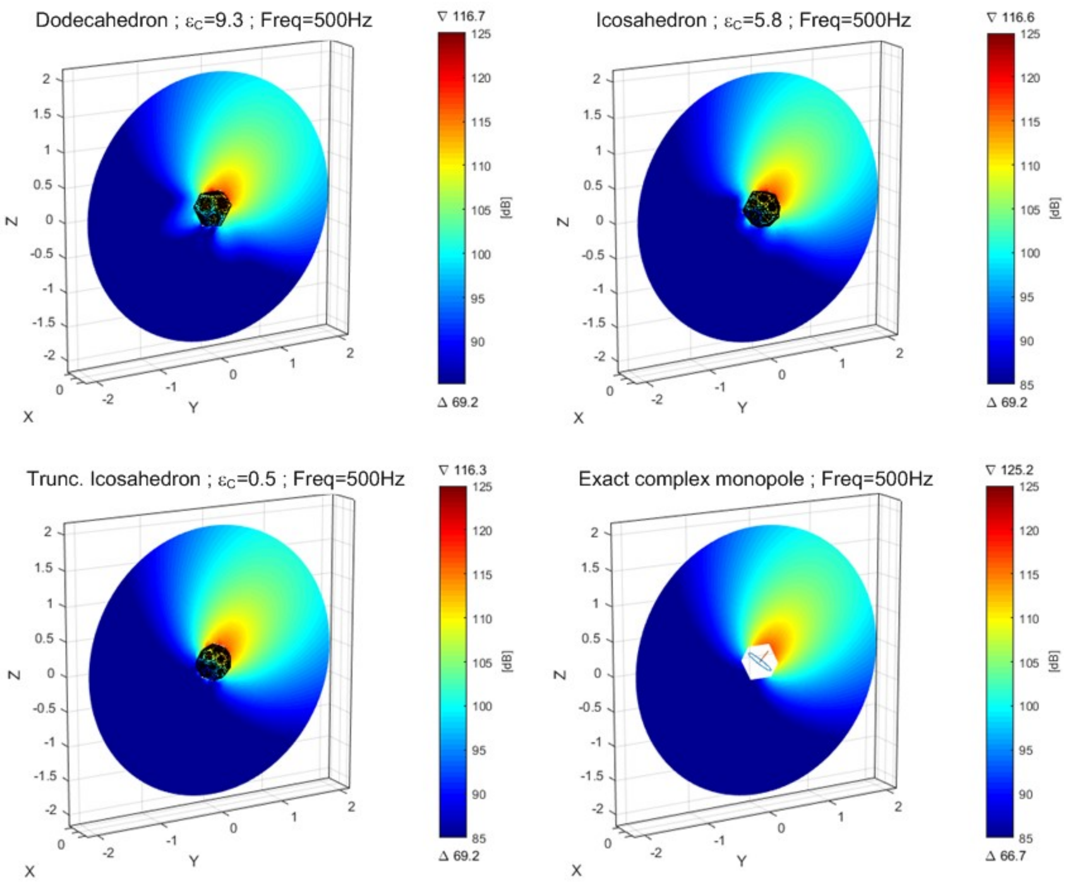

Figure 6 – Simulated and analytical sound field at 500 Hz

Considering the suggested number of loudspeakers per wavelength, the maximum frequency that can be solved by the models would be $f_{rep}=c/2D_L$: 930 Hz, 1250 Hz and 1560 Hz respectively.

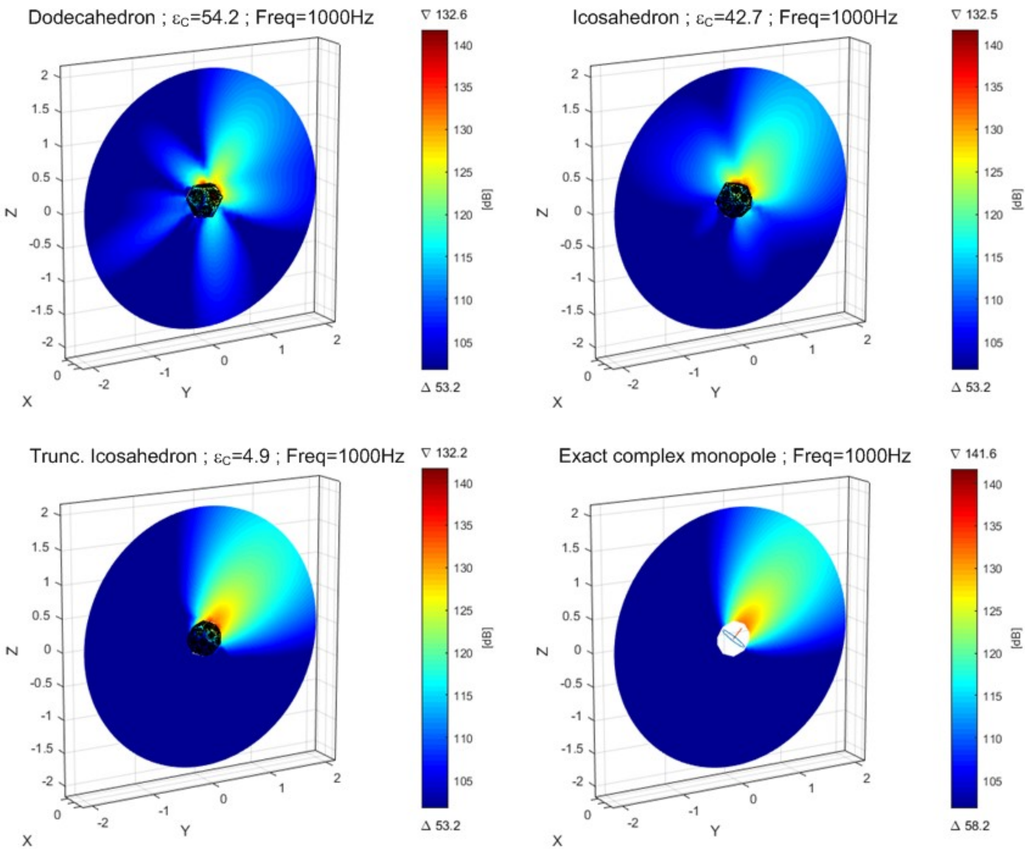

Figure 7 – Simulated and analytical sound field at 1000 Hz

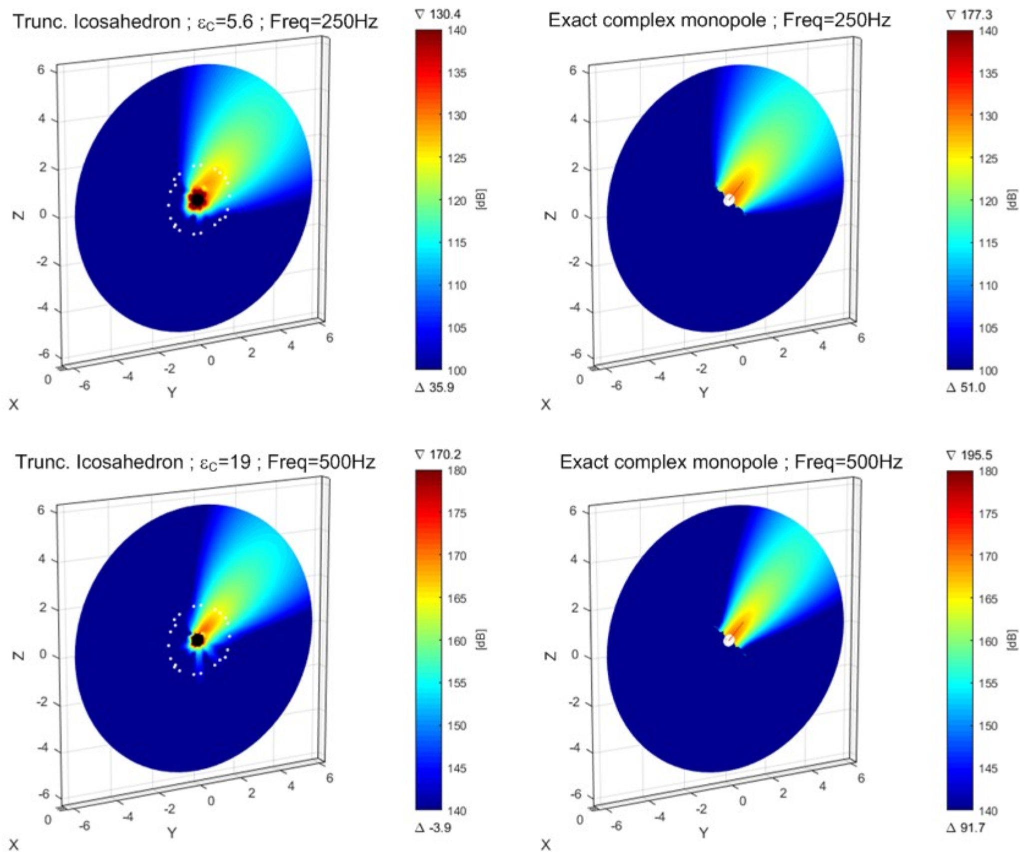

Figure 8 – Simulated and analytical sound field by an extended imaginary part β=1.0 and A_f=3

Fig. 6 and Fig. 7 show the sound radiated by the models and the exact analytical values. At 500 Hz (Fig. 6), the three models are able to reproduce accurately the sound radiation of the complex monopole since the relative errors are smaller than 10%. At 1000 Hz (Fig. 7), only the model with 32 loudspeakers reproduces well the radiation pattern of the monopole with an error ε_C=4.9%.

If the singularity ring of the complex monopole must be always contained inside the model, the size of the model limits the amplitude of the imaginary part of the monopole. If the simulation is performed for bigger imaginary parts: β=0.5 and β=1.0, large errors (ε_C) for all frequencies are obtained. The accuracy of the simulation can be increased by shifting the position of the control points by a factor A_f from their original positions (A_f=1) so that the singularity ring remains inside the surface bounded by the control points (Fig. 8).

6. CONCLUSIONS

The present work shows that the numerical realization of the radiation pattern of complex multipoles with a loudspeaker array is feasible. Accurate simulations can be achieved when the surface of the radiator is equipped by at least two loudspeakers per wavelength. Additionally, the control points must enclose the singularity ring. Highly focused radiation patterns are possible even at low frequencies, but at a certain distance from the radiator, since large imaginary parts are required to produce big singularity rings.

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