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[FEM-BEM applications in vibro-acoustics using GYPSILAB]

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Abstract

The GYPSILAB framework is an open-source MATLAB library which allows the user to solve complex problems using the well-known Finite Element Method (FEM) or the Boundary Element Method (BEM). Cutting edge algorithms are included like a complete $\mathscr{H}$-matrix algebra for the compression of the matrices arising in the BEM. In this paper, we study the strong coupling between an incoming sound wave and a vibrating structure of finite dimension. We discuss its implementation with GYPSILAB. The elastic part is solved using the FEM as it is the most suited for such computations because the medium may be non-homogeneous or anisotropic. The acoustic propagation is computed using the BEM because it is well-suited for infinite media and very accurate. We show that the matrices arising in both methods are assembled and manipulated just like the native MATLAB matrices thanks to operator overloads and only require a minimal implementation effort.

Keywords: FEM-BEM coupling, vibro-acoustics, open-source, GYPSILAB

1 INTRODUCTION

The strong interaction between an acoustic wave and an incoming acoustic wave and an elastic structure is a problem which may arise in multiple fields of engineering. A first example may be the interaction of a sound wave with some reflective or absorbing material in the design-process of concert halls or anti-noise walls. Another example is the propagation of underwater acoustic wave produced by submarine machinery. These waves may go through the hull and may be heard by other submarines, giving away their location.

We study this interaction in the *frequency domain*. In our context, we suppose that the *acoustic medium* is homogeneous and isotropic. This is the case in water or air on reasonably long distances (as long as the medium properties do not change because of the temperature for example). On the other side, the *elastic medium* may be fully inhomogeneous and anisotropic, absorbing, dispersive, etc.

There are many possibilities for the numerical computation of such problems. For complex (multi-)physics problem, the most popular method is probably the Finite Element Method (FEM). It is based on the discretization of the variational formulations associated to the phenomenological equations. The matrices are sparse and can be easily manipulated and inverted thus enabling the resolution of problems with hundreds of millions of unknowns for the most complex cases. The FEM is widely used in the engineering community and may be found in many free or commercial solvers. Unfortunately, the FEM has some major drawbacks. A first one is the high numerical dispersion resulting from the discretization of the domain and the equation. A second drawback, arising in the case of wave propagation in an infinite medium, is the need to mesh a large domain in order to mimic an infinite medium by pushing away the exterior boundary. A third drawback inheriting the second one is the kind of the boundary condition which must be applied on the exterior boundary of the mesh. Once again there are multiple possibilities like an approximation of the Dirichlet-to-Neumann (DtN) operator or the Perfectly-Matched-Layers (PML) which are additional elements with absorbing properties designed to fully absorb the outbound wave without reflection. As a consequence of the assumed complexity of the elastic medium, the FEM is the best choice for the computation of the propagation of an elastic wave.

As mentioned earlier, for FEM, the propagation of a discretized wave induces numerical dispersion increasing with the distance to the *source* [10]. Fortunately, this issue may be circumvented by using the aforementioned BEM [3, 5]. The BEM is based on the discretization of the Boundary Integral Equations related (BIE) to the considered wave equation under the assumption that the propagation medium is homogeneous, isotropic and infinite. They only require the discretization of the *boundary* of the scatterers and the radiation condition is, by construction, taken into account, thus featuring a minimal numerical dispersion. Of course, the BEM also have major drawbacks. For example, they require the numerical computation of weakly singular integrals. The most limiting one is that the arising matrices are *dense* and *non-symetric* as the BIEs are inherently convolution products. In order to compute problems featuring more than a few hundred thousands degrees of freedom, it is mandatory to compress these matrices. Existing methods are the Fast Multipole Method [7], the hierarchical matrices ($\mathcal{H}$-matrices) [8], and more recently the Sparse Cardinal Sine Decomposition (SCSD) [6]. These difficulties are one of the reasons why the BEM is much less popular. The coupling between FEM and BEM is even less popular.

This paper is organized as follows. First we recall the well-known elasticity problem. In the second part, we recall the BEM and we justify the choice of our formulation. In the third part, we introduce the GYPSILAB framework, we give the coupling condition between the elastic medium and the acoustic medium and we deduce the variational formulations which will be discretized. We describe the key points for the implementation using GYPSILAB and the resolution procedure. Finally, we give a validation example.

2 ELASTIC WAVE

In the following, we assume that the elastic medium is homogeneous and isotropic as all the following development remain valid with little changes.

Let $\Omega \subset \mathbb{R}^d, d = 2,3$ the elastic medium and Γ its boundary with *outbound* normal $\mathbf{n}$, we aim at computing the *displacement* $\mathbf{u} = [u_1, \ldots, u_d]^T$.

Let ω the angular frequency of the incoming acoustic wave, the corresponding elastic-wave equation *in the frequency domain* in the absence of source terms is, following [9],

$$\mathrm{div}(\sigma(\mathbf{u})) + \rho_S \omega^2 \mathbf{u} = 0, \tag{1}$$

where ρ_S is the density of the material. The *constraint tensor* $\sigma(\mathbf{u})$ reads as

$$\sigma(\mathbf{u}) = \mu \left(\nabla \mathbf{u} + \nabla^T \mathbf{u} \right) + \lambda \left(\nabla \cdot \mathbf{u} \right) \mathbb{I}, \tag{2}$$

where (λ, μ) are the Lamé coefficients and $\mathbb{I}$ is the identity matrix. Please note that this equation may be fully re-written by making the longitudinal (c_L) and shear wave (c_T) celerities appear by setting

$$\mu = \rho_S \cdot c_T^2 \qquad \text{and} \qquad \lambda = \rho_S \cdot (c_L^2 - 2c_T^2). \tag{3}$$

In the context of absorbing media, c_T and c_L are complex.

3 BOUNDARY INTEGRAL EQUATIONS FOR THE HELMHOLTZ EQUATION

We aim at computing the pressure field p in some propagation domain. We re-use the notations introduced in section 2. The pressure field may be computed by solving the Helmholtz equation, e.g. [5],

$$-\Delta p - k_0^2 p = 0 \quad + \quad \text{boundary conditions on } \Gamma, \tag{4}$$

in the whole propagation domain, where $k_0 = \omega / c_0$ is the *wave number* and c_0 is the celerity of sound wave in the propagation medium. The field p must additionaly obey the Sommerfeld radiation condition

$$\lim_{r \to \infty} |r|^{(d-1)/2} (\partial_r p - i k_0 p) = 0, \tag{5}$$

where r is the distance to the origin and d the dimension of the propagation medium.
There exists particular solutions of (4) called *fundamental solutions* or *Green kernels* G, see [3, 5]. They are

$$G(x,y) = \frac{i}{4}H_0^{(1)}(k|x-y|) \quad \text{for } d=2 \qquad \text{and} \qquad G(x,y) = \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|} \quad \text{for } d=3, \tag{6}$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order 0.

Remark 1. *Close to $r \to 0$, $H_0^{(1)}(kr)$ behaves like $-1/(2\pi)\log(r)$.*

By multiplying (4) by G, then integrating by parts, using the Cauchy formula and the radiation condition (5), one can show that there exist λ and μ such that we have the following *integral representation formula*

$$p(x) = \int_\Gamma G(x,y)\mu(y)\,d\gamma_y - \int_\Gamma \frac{\partial G}{\partial n_y}(x,y)\lambda(y)\,d\gamma_y, \qquad \forall x \notin \Gamma. \tag{7}$$

Remark 2. *For given λ and μ, p computed using (7) is a solution of (4) obeying (5).*

Boundary integral equations are obtained by applying the trace operator to (7). The first step is to decompose the total field $p^{\mathrm{tot}} = p^{\mathrm{i}} + p^{\mathrm{s}}$ where p^{i} is the known incident field and p^{s} is the unknown *scattered* field. It is not required for p^{i} to follow (5); this is the case when it is a plane wave $p^{\mathrm{i}}(x) = e^{i\vec{k}_0\cdot x}$.
For example, the scattering by a sound-hard obstacle (homogeneous Neumann boundary condition on Γ) can be computed by solving the following equation, see for example [5],

$$\frac{\partial p^{\mathrm{i}}}{\partial n}(x) = \int_\Gamma \frac{\partial^2 G}{\partial n_x \partial n_y}(x,y)\,\lambda(y)\,d\gamma_y, \qquad x \in \Gamma. \tag{8}$$

One way to solve equation (8) can be solved by using a Galerkin formulation which reads: find λ such that for any complex-valued test function w

$$\int_\Gamma \bar{w}(x)\frac{\partial p^{\mathrm{i}}}{\partial n}(x)\,d\gamma_x = k_0^2\int_{\Gamma\times\Gamma} \bar{w}(x)G(x,y)\lambda(y)(\mathbf{n}_x\cdot\mathbf{n}_y)\,d\gamma_x d\gamma_y - \tag{9}$$

$$\int_{\Gamma\times\Gamma} (\mathbf{n}_x \times \nabla_\Gamma \bar{w}(x))\cdot(\mathbf{n}_y \times \nabla_\Gamma \lambda(y))\,G(x,y)\,d\gamma_x d\gamma_y. \tag{10}$$

Some advantages of the BEM appear immediately. In particular, one only need a discretization of Γ and there is no need to enforce the radiation condition as it is *intrinsinc* !
In general, the systems arising in the BEM are very ill-conditioned and ill-posed for certain frequencies. Fortunately, some formulations circumvent these limitations at the cost of an increased assembling time. In the following, we will use the Brakhage & Werner [4] representation formula for the scattered field which reads

$$p^{\mathrm{s}}(x) = \int_\Gamma \left(G(x,y) + i\beta\frac{\partial G}{\partial n_y}(x,y)\right)\alpha(y)\,d\gamma_y, \qquad x \notin \Gamma, \tag{11}$$

$$\frac{\partial p^{\mathrm{s}}}{\partial n}(x) = \int_\Gamma \left(\frac{\partial G}{\partial n_x}(x,y) + i\beta\frac{\partial^2 G}{\partial n_x \partial n_y}(x,y)\right)\alpha(y)\,d\gamma_y, \tag{12}$$

or for $x \in \Gamma$

$$p^{\mathrm{s}}(x) = \int_\Gamma \left(G(x,y) + i\beta\frac{\partial G}{\partial n_y}(x,y)\right)\alpha(y)\,d\gamma_y + i\beta\frac{\alpha(x)}{2}, \qquad x \in \Gamma, \tag{13}$$

$$\frac{\partial p^{\mathrm{s}}}{\partial n}(x) = -\frac{\alpha(x)}{2} + \int_\Gamma \left(\frac{\partial G}{\partial n_x}(x,y) + i\beta\frac{\partial^2 G}{\partial n_x \partial n_y}(x,y)\right)\alpha(y)\,d\gamma_y, \tag{14}$$

where β is a parameter whose optimal value is $\beta = \min(1/k_0, 2)$ (see [4]) and α some unknown. For convenience, we write

$$p^{\mathrm{s}}(x) = \mathrm{BW}\alpha \qquad \text{and} \qquad \frac{\partial p^{\mathrm{s}}}{\partial n}(x) = \frac{\partial \mathrm{BW}}{\partial n}\alpha, \qquad x \in \Gamma. \tag{15}$$

We insist on the fact that α does *not* need to hold a physical signification as the value of interest is p^{s} which can be computed in the whole domain using the Brakhage & Werner representation formula (11).

4 COUPLING, VARIATIONAL FORMULATIONS AND RESOLUTION

4.1 A few words about GYPSILAB

GYPSILAB is an open-source MATLAB prototyping framework available at [1] under the GPL3.0 licence. It aims at simplifying the development of numerical methods for multiphysics problems. It focuses mainly on the FEM and BEM formulations.

The matrices arising in such methods can be computed thanks to multiple overloads of the base `integral()` function of MATLAB whose return type depends on the input parameters. In the case of the FEM, the matrices are automatically sparse. When dealing with BEM, the matrices may be full (in that case the user should be *very* careful if the number of unknowns is too high) or in the hierarchical format ($\mathscr{H}$-matrices). In fact, $\mathscr{H}$-matrices in GYPSILAB can be considered as an *extension to the* MATLAB *language* since a complete algebra has been implemented: matrix-vector product, multiplication by a scalar, matrix-matrix multiplication, addition, addition with a sparse matrix, concatenation, LU-decomposition, etc. Consequently, the user does not need to know which type he is currently manipulating and the coding effort is minimal. Other possibilities offered by the software are basic mesh manipulations, mesh refinement with `MMG` [11], and ray-tracing. Many examples are available in the `./nonRegressionTests/` subdirectory of the main GYPSILAB directory. For more details, we refer to [2].

4.2 Coupling

The coupling between the elastic problem and the acoustic problem is done easily by enforcing two continuity condition at the interface Γ between the two domains. First we enforce the continuity of the displacement at the boundary. which reads, extrapolating from [5] (2.1.1) p. 9,

$$\rho_0 \omega^2 (\mathbf{u} \cdot \mathbf{n}) = \frac{\partial p^{\mathrm{tot}}}{\partial n}, \tag{16}$$

where ρ_0 is the density of the acoustic medium. Second we enforce the continuity of the constraint which reads

$$\sigma(\mathbf{u}) \cdot \mathbf{n} + p^{\mathrm{tot}} \mathbf{n} = 0. \tag{17}$$

Next, we introduce the test function $\mathbf{v}$ for the elasticity problem and w the test function for the acoustic problem. By testing (16) with w and (17) and (1) with $\mathbf{v}$, we obtain the following coupled system of equations in the Galerkin framework

$$\int_\Omega \left(\sigma(\mathbf{u}) : \nabla \bar{\mathbf{v}} - \rho_s \omega^2 \mathbf{u} \cdot \bar{\mathbf{v}}\right) d\Omega + \int_\Gamma (\bar{\mathbf{v}} \cdot \mathbf{n})(\mathrm{BW}\alpha)\, d\gamma = -\int_\Gamma (\bar{\mathbf{v}} \cdot \mathbf{n}) p^{\mathrm{i}}\, d\gamma, \tag{18}$$

$$\int_\Gamma \bar{w}(\mathbf{u} \cdot \mathbf{n})\, d\gamma - \int_\Gamma \bar{w} \left(\frac{\partial \mathrm{BW}}{\partial n}\alpha\right) d\gamma = \int_\Gamma \bar{w} \frac{\partial p^{\mathrm{i}}}{\partial n}\, d\gamma. \tag{19}$$

This is a 2×2 block-system of equations such that

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{u} \\ \alpha \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{20}$$

where the unknowns are $\mathbf{u}$ (displacement in Ω, vector) and α (defined on Γ only, no physical signification, scalar). A_{11} is the pure FEM block which admits a sparse discretization. A_{22} is the pure BEM block which admits a full or hierarchical representation. The coupling term A_{21} is sparse while A_{12} can be either full or in the $\mathscr{H}$-matrix format.

We choose to solve the block-system of equations (20) using a Schur-complement. The acoustic-unknown α is solution of

$$\left(A_{22} - A_{21} \cdot A_{11}^{-1} \cdot A_{12}\right) \alpha = b_2 - A_{21} \cdot A_{11}^{-1} \cdot b_1, \tag{21}$$

which is solved iteratively with the GMRES algorithm using the LU-decomposition of A_{22}.

4.3 Some aspects of the implementation with GYPSILAB

In this part, we describe some aspects of the implementation of the coupled FEM-BEM problem within the GYPSILAB framework. Unless specified otherwise, the names of the variables in the code reflect the name given previously and we assume that the material properties have already been defined. For the sake of simplicity, we solve the *two-dimensional* scattering problem.

Pre-processing The first step is to obtain the mesh (in our case it can be generated using GYPSILAB) and extract its boundary, define the integration domain and the finite element spaces. This is done with

```
% mesh
mesh = mshSquare(N, [L, e]); % N: number of nodes, [L, e]: dimensions
bnd  = mesh.bnd;             % boundary of mesh
% integration domain
Omega = dom(mesh, 3); % integration over mesh, 3 quadrature nodes
Gamma = dom(bnd, 3);  % integration over bnd, 3 quad. nodes
% finite element spaces
U = fem(mesh, 'P1'); % linear functions
V = fem(mesh, 'P1');
a = fem(bnd, 'P1');
w = fem(bnd, 'P1');
```

Assuming the incident wave is a plane wave, it is defined as

```
Ui = @(X) exp(1i*k0*X*X0');           % plane wave
gxUi{1} = @(X) 1i*k0*X0(1) .* Ui(X); % 1st component of gradient
gxUi{2} = @(X) 1i*k0*X0(2) .* Ui(X);
```

where `X0` is the direction of propagation.

Assembling operators In order to build the BEM matrices, the user needs to define the Green kernels. Fortunately, some are already implemented.

```
% Green kernel in 2D
Gxy = @(X,Y) femGreenKernel(X,Y,'[H0(kr)]', k0);
% gradient components of the Green kernel
gyGxy{1} = @(X,Y) femGreenKernel(X,Y,'grady[H0(kr)]1', k0);
gyGxy{2} = @(X,Y) femGreenKernel(X,Y,'grady[H0(kr)]2', k0);
```

One important thing to know is that GYPSILAB does *not* know wether an unknown or test function is a vector or a scalar field. Consequently, integrals which do not involve uniquely the scalar product of two vector fields should be expanded component by component. For example, the integral $\int_\Gamma (\bar{\mathbf{v}} \cdot \mathbf{n}) p^{\mathrm{i}}\, d\gamma$ becomes $\int_\Gamma \bar{v}_1 n_1 p^{\mathrm{i}}\, d\gamma + \int_\Gamma \bar{v}_2 n_2 p^{\mathrm{i}}\, d\gamma$

```
rhsFEM = -1i/4*(integral(Gamma,ntimes(V,1),Ui) + ...
  integral(Gamma,ntimes(V,2),Ui));
```

In the computation of the full-FEM block, we need to compute integrals of the form $\mu \cdot \int_\Omega \partial_1 v_2 \cdot \partial_2 u_1$ which can be implemented easily like

```
mug1g2 = mu*integral(Omega,grad(V,1),grad(U,2));
```

For the BEM matrices, we use once again the `integral()` function. By passing a tolerance parameter `tol`, the matrix returned is in the compressed $\mathscr{H}$-matrix format. When assembling the pure acoustic block containing (14), one needs to assemble at some point the variational formulation (9). Since the integrals are singular and `integral()` integrates using a Gauss quadrature, we must regularize it by removing the singularity integrated by quadrature and replacing it with a semi-analytic formula. This is done by calling the `regularize()` function which returns a sparse matrix. In our example, it reads

```
A22 = 1i/4* ...
  (k0^2*integral(Gamma,Gamma,ntimes(w),Gxy,ntimes(a),tol) - ...
  integral(Gamma,Gamma,nxgrad(w),Gxy,nxgrad(a),tol));
A22 = A22 - 1/(2*pi)* ...
  (k0^2*regularize(Gamma,Gamma,ntimes(w),'[log(r)]',ntimes(a)) - ...
  regularize(Gamma,Gamma,nxgrad(w),'[log(r)]',nxgrad(a)));
```

where `A22` is a $\mathscr{H}$-matrix. When `tol` is removed, the result is a full matrix.

The source code for the complete construction of the different matrices and right hand sides is available in the corresponding folder in `./nonRegressionTest/` within the GYPSILAB directory. It covers many of the possibilities covered by the framework.

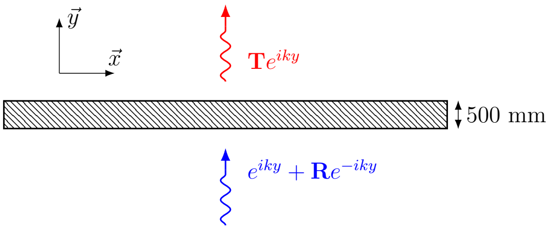
Solving the problem The Schur complement (21) is implemented as follows

```
A11 = decomposition(A11); % decomposition of A11
rhs = b2 - A21*(A11\b1);
[LL,UU] = lu(A22);          % lu of H-matrix A22
prec = @(V) UU\(LL\V);      % definition of preconditioner
lhs  = @(V) A22*V - A21*(A11\(A12*V)); % matrix-vector product
alpha = gmres(lhs,rhs,[],tol,100,prec); % compute solution
```

The Brakhage-Werner formulation being well-conditioned, the use of the preconditioner may not be mandatory.

Post-processing In the previous step, we computed the unknown α. We assume that the post-processing consists in computing the total field at a given point x_s in the propagation domain sufficiently far from the boundary so that there is no need for ragularizing. Other kinds of post-processing could be carried out in the same fashion. We will use the representation formula (11) implemented by first computing the "radiation matrix" `rad`, then using it to compute the scattered field `ps`. Finally, we add the incident field to obtain the total field `ptot`. The implementation is as follows

```
rad  = 1i/4*(integral(Xs,Gamma,Gxy,a) + ...
    1i*beta*integral(Xs,Gamma,gyGxy,ntimes(a)));
ps   = rad*alpha;    % scattered field at Xs
ptot = ps + Ui(Xs);  % total field at Xs
```



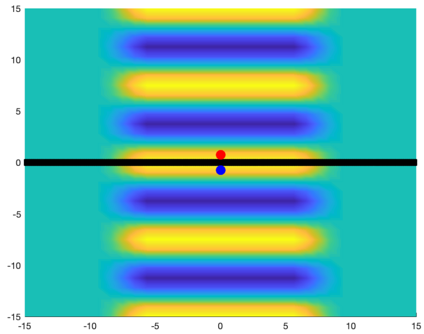


Figure 1. Left: Setting for the numerical computation of the transmision and reflexion coefficients $\mathbf{T}$ and $\mathbf{R}$. Right: cut-off function for the real part of the incident plane wave associated to the measurement points (blue for reflexion and right for transmission).

5 NUMERICAL EXAMPLE

We present here a validation example available with GYPSILAB in 2D and 3D. The goal is to compute de reflexion and transmission coefficients $\mathbf{R}$ and $\mathbf{T}$ for a three-layered medium: the incoming wave propagates through three successive media. A part is reflected and transmitted at each interface. By definition, for a normalized incident wave, the transmission coefficient is the amplitude of the total field in the third medium and the reflexion coefficient is the amplitude of the backward-propagating wave in the first medium. In our setting, we suppose moreover that the first and the third medium have the *same physical properties*.

Of course, the theoretical problem is purely one-dimensional and the purpose of our study is to show that one can recover the analytical values by solving a two-dimensional problem for a slab with the correct hypotheses.

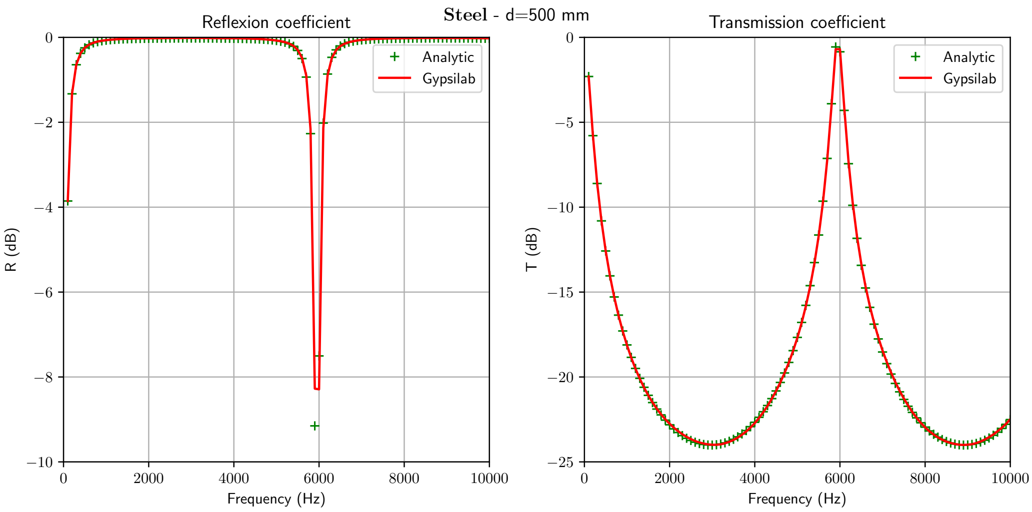


Figure 2. Top left: scattered field at 10000 Hz (used to compute the reflexion coefficient). Top right: Total field at 10000 Hz (used to compute the transmission coefficient). Bottom: Reflexion and transmission coefficients for a steel plate with thicknes $d = 500$ mm in the range $100 - 10000$ Hz.

First we set the thickness of the slab and a frequency. The length of the plate is a fixed multiple of the wavelength. In order to avoid acoustic scattering at the corners of the slab, we localize the plane wave by multiplying it with a cut-off function centered on the middle of the slab in the propagating direction whose bandwidth is significantly smaller than the width of the slab but is also a fixed multiple of the wavelength. We also set the shear-wave celerity $c_T = 0$ (see (3)) since we forbid any displacement along the longest dimension of the slab (direction $\vec{x}$ on Fig. 1). The exterior medium is water ($c_0 = 1500$ m.s^{-1} and $\rho_0 = 1000$ kg) and the elastic medium is steel ($c_L = 6000$ m.s^{-1} and $\rho_s = 8000$ kg).

The coefficients **T** and **R** are computed in the range $f \in [100, 10000]$ Hz by probing the total field at a distance one wavelength above the slab, resp. one wavelength below. The results are represented on Fig. 2. We observe that the analytical computation and the GYPSILAB computation overlap thus validating our setting.

6 CONCLUSIONS

In this paper, we showed how one can efficiently implement strongly coupled FEM-BEM problems by combining the powerful high-level interfaces provided by both the native MATLAB API and the GYPSILAB framework. We demonstrated that the variational formulations can be easily implemented and the development time is short making it a prime framework for fast prototyping. The three-dimensional case has also been validated but requires much more computational time as the number of degrees of freedom increases much faster with the frequency than in the two-dimensional case. It is not presented in the context of this paper as we lack space but the implementation is done similarly.

In the future, this work will be applied to more complex structures of research or industrial dimension. Some of our topics of interest are the design of music instruments, acoustics of underwater structure or non-homogeneous anisotropic media.

REFERENCES

[1] `https://github.com/matthieuaussal/gypsilab`

[2] Alouges, F.; Aussal, M. FEM and BEM simulations with the GYPSILAB framework, SMAI-JCM, Vol. 4, 2018, pp. 297–318

[3] Colton, D. ; Kress, R. Integral Equation Methods in Scattering Theory, SIAM Classiccs in Applied Mathematics, 2013

[4] Kress, R Minimizing the condition number of boundary integral operators in acoustic and electromagnetic scattering, Q. J. Mech. Appl. Math., Vol. 38 (2), 1985, pp. 323–341

[5] Nédélec, J.-C. Acoustic and Electromagnetic Equations, Integral Representations for Harmonic Problems, Springer Applied Mathematical Sciences 144, 2001

[6] Alouges, F.; Aussal, M. The Sparse Cardinal Sine Decomposition and its Application to fast numerical convolution, Numerical Algorithms, Vol. 70 (2), 2015, pp. 427–448

[7] Greengard, L. The rapid evaluation of potential fields in particle systems, MIT Press, 1988

[8] Börm, S.; Grasedyck, L.; Hackbusch, W. Hierarchical Matrices, technical. report, 2015

[9] Skelton, E. A.; James, J. H. Theoretical Acoustics od Underwater Structures, Imperial College Press, 1997

[10] Terrasse, I.; Abboud, T. Modélisation des phènomènes de propagation d'onde, `https://www.math.u-bordeaux.fr/ abachelo/Abboud-Terrasse.pdf`

[11] `https://www.mmgtools.org/`