PROCEEDINGS of the
23rd International Congress on Acoustics
9 to 13 September 2019 in Aachen, Germany

Analytic solutions for acoustic scattering by blade rows with complex boundaries: porosity, compliance and impedance

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Abstract

The acoustic scattering by an infinite cascade of flat plates is analysed for a range of boundary conditions including porosity, compliance and impedance. By employing the Wiener-Hopf method, analytic expressions are derived for the pressure field that are uniformly valid throughout the entire domain. This enables exact solutions for the unsteady lift, sound power output and other acoustic quantities. The Wiener-Hopf analysis shows that complex boundaries have a significant effect on the modal structure of the cascade's inter-blade region, whilst the modes in the far-field regions are unchanged. The results demonstrate the considerable impact of small modifications to the boundary conditions and highlight the importance of the accurate modelling of aeroelastic effects when predicting sound transmission and generation in turbomachinery.

Keywords: fluid-structure interaction, turbomachinery noise, Wiener-Hopf method

1 INTRODUCTION

Turbomachinery noise remains a significant contributor to overall aero-engine noise [12]. A considerable source of broadband noise is the so-called "rotor-stator interaction", where unsteady wakes shed by rotors interact with downstream stators. Much progress has been made in understanding rotor-stator interaction noise when the blades are modelled as flat plates [11, 6, 13] or with realistic geometries [2]. However, research into blades with complex boundaries – where the blades are not necessarily stationary and rigid – are far less developed, especially from an analytic standpoint. Consequently, in the present work we derive analytic solutions for the acoustic scattering by blade rows where the boundary conditions are not limited to a typical rigid no-flux boundary condition. The method is capable of modelling a range of boundary conditions of practical interest, some of which are detailed below.

Aspirations for lightweight and efficient engines have driven the design of thinner and lighter blades in turbomachinery [14]. As a result, aeroelastic effects such as flutter and resonance must be considered in modern turbomachinery design and testing. The rapid and accurate prediction of the aeroacoustic performance of turbomachinery with consideration of aeroelastic effects is therefore essential in evaluating the appropriateness of potential blade designs. Analytic solutions are excellent candidates for this task [6, 13, 2], but are presently limited to rigid blades with no consideration of aeroelastic effects. The present study permits the analysis of compliant blades [5], where the plate deforms with a local response to the pressure gradient across the blade. More general impedance relations with background flow [10] are also amenable to the present approach.

An influential trend in aeroacoustic research is to modify aerofoils with noise reducing technologies. A popular choice is a poroelastic extension, which was originally inspired by the silent flight of owls [7]. These extensions have been applied to semi-infinite [8] and finite [1] plates and have demonstrated considerable noise reductions. The effects of porosity on blades in cascade formations has not received any analytical treatment, except for recent work by the authors which showed that aerodynamic losses can be mitigated by modifying the blade spacing appropriately [3]. The approach of the present research permits aeroacoustic analysis of porous blades under the assumption of a Darcy-type condition where the seepage velocity through the blade is proportional to the pressure jump across the blade.

In the present research, we extend the work of Glegg [6] and Posson [13] to analyse the acoustic scattering by a cascade of flat plates with a range of boundary conditions using the Wiener-Hopf method. A significant advantage of the presented technique is that the method is identical regardless of the boundary condition – the

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only effects of modifying the boundary condition are to modify the kernel in the Wiener-Hopf method. We consider four possible boundary conditions labelled cases 0–III. Physically, case 0 corresponds to rigid blades; case I, porous or compliant blades with no background flow; case II, Darcy flow with background flow; and case III, a general impedance relation. Mathematically, case 0 corresponds to a Neumann boundary; case I, a Robin boundary; case II, an oblique derivative boundary; and case III, a generalised Cauchy boundary.

We begin by presenting a mathematical model for the blade row in section 2, including the modelling of the various boundary conditions. We then present some details of the mathematical solution in section 3. In section 4 we present a range of results including the unsteady lift, sound power output, and scattered pressure field. Finally, in section 5 we summarise our work and suggest future directions of research.

2 MATHEMATICAL FORMULATION

We consider a rectilinear cascade of blades in a uniform, subsonic flow as illustrated in figure 1. As is typical in these analyses [6, 11], it is useful to rotate the coordinate system so that

$$(x^*, y^*, z^*) = (\tilde{x}\cos(\chi^*) - \tilde{y}\sin(\chi^*), \tilde{x}\sin(\chi^*) + \tilde{y}\cos(\chi^*), \tilde{z}),$$

and the x^* and y^* coordinates are tangent and normal to the blades respectively. The background flow is tangent

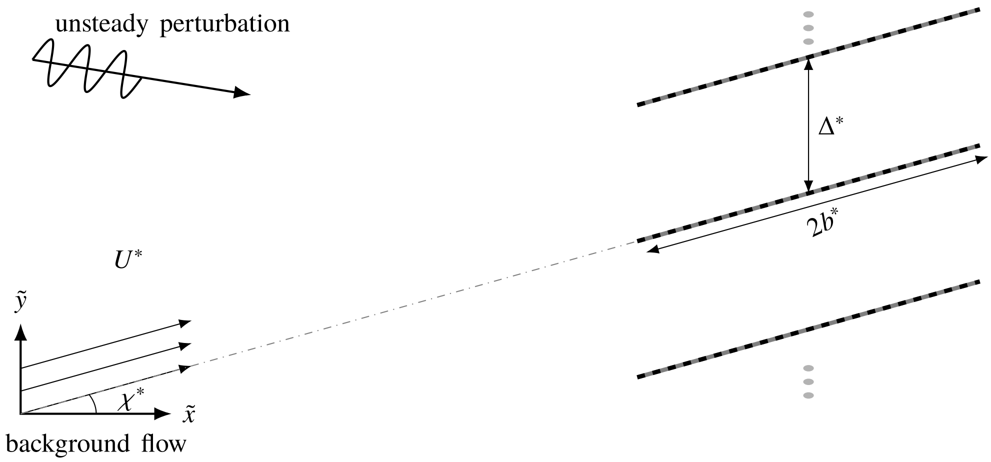

Figure 1. A rectilinear cascade of flat plates with complex boundaries.

to the blades, and may have a spanwise component such that $\boldsymbol{U}^* = (U^*, 0, W^*)$. The blades in the cascade have dimensional length $2b^*$ and are inclined at stagger angle χ^*. The distance between adjacent blades is Δ^*. We assume that a vortical or acoustic wave is incident on the cascade, resulting in a velocity perturbation $\boldsymbol{u}^*$ to the mean flow. The Kutta condition is satisfied by ensuring that there is no pressure jump across the wake.

We introduce an acoustic potential function for the scattered field defined by

$$\nabla\phi^* = \boldsymbol{u}^*.$$

so that conservation of mass yields the convected Helmholtz equation

$$\frac{1}{c_0^2}\frac{D_0^2\phi^*}{D_0 t^{*2}} - \nabla^2\phi^* = 0. \tag{1}$$

where c_0 is the isentropic speed of sound. We suppose that the unsteady perturbation incident on the cascade takes the form

$$\phi_i^* = A^* \exp\left[\mathrm{i}\left(k_x^* x^* + k_y^* y^* + k_z^* z^* - \omega^* t^*\right)\right]. \tag{2}$$

Since the system is infinite in the spanwise direction, the scattered solution $\boldsymbol{u}^*$ must also have harmonic dependence in the z-direction. Accordingly, making the following convective transformation and appropriate non-dimensionalisations

$$\phi^*(x^*, y^*, z^*, t^*) = U^* b^* \phi(x, y) \exp\left[\mathrm{i}\omega\left(-M^2\delta x + z - t\right)\right],$$

$$\omega = \frac{b^*}{U^*}(\omega^* - W^* k_z^*), \qquad w^2 = (M\delta)^2 - (k_z/\beta)^2 - (2 + 2\mathrm{i})k_z\delta M^2 W(1 - Wk_z),$$

reduces (1) to

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \omega^2 w^2\right)\phi = 0. \tag{3}$$

All undefined quantities used in this non-dimensionalisation are defined in [9]. The blades now have horizontal spacing d and vertical spacing s.

2.1 Boundary conditions

We now introduce the boundary conditions for the problem. It is sufficient to specify the behaviour along $y = ns$ for $n \in \mathbb{Z}$. In the following analysis, we use Δ_n and Σ_n to denote the difference and sum of a given quantity either side of the n^{th} blade and wake.

2.1.1 Upstream boundary condition

There may be no discontinuities upstream of the blade row. Consequently, we write

$$\Delta_n\phi(x) = 0, \qquad \phi < nd. \tag{4}$$

2.1.2 Blade boundary condition

Several possible boundary conditions that can be modelled with the present approach. Mathematical analysis reveals that physically relevant boundary conditions can be expressed in terms of ϕ and its derivatives. Accordingly, we may characterise all the possible modified boundary conditions in the general form

$$\begin{aligned}\Sigma_n\left[\frac{\partial\phi}{\partial y}\right](x) &= -2w_0 \exp\left[\mathrm{i}k_x(\delta(nd + x) + k_y ns)\right] \\ &\quad + \mu_0\Delta_n[\phi](x) + \mu_1\Delta_n[\phi_x](x) + \mu_2\Delta_n[\phi_{x,x}](x), \qquad nd < x < nd + 2,\end{aligned} \tag{5}$$

where the μ_n are summarised in table 1 for the different boundary conditions. Furthermore, in the present analysis we do not allow any added mass and therefore enforce that there is no jump in the normal velocity either side of the plate. Accordingly, we write

$$\Delta_n\left[\frac{\partial\phi}{\partial y}\right](x) = 0, \qquad nd < x < nd + 2. \tag{6}$$

2.1.3 Downstream boundary conditions

Downstream we require the pressure jump across the wake to vanish so that

$$\Delta_n[\phi](x) = 2\pi\mathrm{i}P\exp[\mathrm{i}\omega\delta x], \qquad x > nd + 2, \tag{7}$$

where P is a constant of integration that will be specified by enforcing the Kutta condition.

Case	Model	μ_0	μ_1	μ_2
Case 0 [6, 13]	rigid, impermeable	0	0	0
Case I [5, 8, 9]	porous, compliant (no background flow)	C_I	0	0
Case II	porous, compliant (background flow)	$\mathrm{i}\omega\delta C_{II}$	$-C_{II}$	0
Case III [10]	impedance	$-2\omega^2(1+Wk_z)C_{III}$	$-2\mathrm{i}M^2\omega C_{III}$	C_{III}

Table 1. Summary of possible boundary conditions and corresponding μ_0, μ_1 and μ_2 values for equation (5). The C_I, C_{II} and C_{III} each represent constants that control the porosity, compliance or impedance.

Additionally, the normal velocity across the wake must vanish, i.e.

$$\Delta_n\left[\frac{\partial\phi}{\partial y}\right](x) = 0, \qquad x > nd + 2. \tag{8}$$

This completes the description of the mathematical model.

3 SOLUTION

We now present the mathematical solution to the Helmholtz equation (3) subject to the boundary conditions (4), (5), (6), (7) and (8).

As is typical in cascade acoustics problems, we employ integral transforms to obtain a solution that is uniformly valid [11, 6, 13]. However, ϕ is discontinuous across each blade and wake in the y-direction. Therefore, $\partial\phi/\partial y$ possesses non-integrable singularities thus preventing the application of a Fourier transform. Consequently, we must regularise the derivatives of ϕ to remove these non-integrable singularities. To this end, we introduce introduce generalised derivatives and write

$$\frac{\partial^2\phi}{\partial y^2} = \frac{\tilde{\partial}^2\phi}{\tilde{\partial} y^2} - \sum_{n=-\infty}^{\infty}\Delta_n\left[\phi\right](x)\delta'(y-ns) - \sum_{n=-\infty}^{\infty}\Delta_n\left[\frac{\partial\phi}{\partial y}\right](x)\delta(y-ns), \tag{9}$$

where $\tilde{\partial}$ represents the partial derivative with discontinuities removed. The second term in (9) vanishes because we do not allow added mass so there is zero jump in normal velocity across the blade (6) and wake (8).

The scattered solution must obey the same quasi-periodicity relation as the incident field (2). Consequently, the scattered acoustic potential function in the entire plane may be reduced to a single channel in the domain by writing

$$\phi(x+nd, y+ns) = \phi(x,y)\mathrm{e}^{\mathrm{i}n\sigma'}, \tag{10}$$

where the inter-blade phase angle for ϕ is $\sigma' = k_x(\delta d + k_y s)$. Substituting (9) into the Helmholtz equation (3) and applying the phase angle relation (10) yields

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \omega^2 w^2\phi = \sum_{n=-\infty}^{\infty}\Delta_0\left[\phi\right](x-nd)\delta'(y-ns)\mathrm{e}^{\mathrm{i}n\sigma'}. \tag{11}$$

Reference	gap-to-chord ratio $\Delta/2$	stagger angle χ	Mach number M	reduced frequency ω	inter-blade phase angle σ
Glegg [6]	0.6	40	0.3	0–30	$3\pi/4$
Posson [13]	sec (20°)	20°	0.2	5	$3\pi/4$

Table 2. Summary of parameters used in results section. These cases both considered a convected gust.

Following Glegg [6], we Fourier transform in the x- and y-directions, and invert the Fourier transform in the y-direction to obtain

$$\Sigma_0 \left[\frac{\partial \phi}{\partial y}\right](x) = -4\pi \int_{-\infty}^{\infty} D(\gamma) j(\gamma) e^{-i\gamma x} d\gamma, \tag{12}$$

where j is defined in [6]. The problem is now to find $D(\gamma)$ which represents the Fourier transform of the jump in acoustic potential either side of the blade and wake. We now solve equation (12) subject to the remaining boundary conditions applied on $y = 0$ (4,5,7). Consequently, we have an integral equation (12) subject to mixed value boundary conditions (4,5,7). We solve this system with the Wiener-Hopf method using a similar approach to Glegg [6], and the solution for D is given by

$$\begin{aligned} D(\gamma) =& \frac{w_0}{(2\pi)^2 i(\gamma + \delta k_x) K_-(-\delta k_x) K_+(\gamma)} + \frac{w_0 \delta(\omega - k_x) e^{2i(\gamma + \delta k_x)}}{(2\pi)^2 i(\gamma + \delta k_x)(\gamma + \delta\omega) K_+(-\delta k_x) K_-(\gamma)} \\ & - \sum_{n=0}^{\infty} \frac{(\mathcal{A}_n + C_n)\, e^{2i(\gamma - \theta_n^-)}}{i(\gamma + \delta\omega)(\gamma - \theta_n^-)} \cdot \frac{K_-(\theta_n^-)}{K_-(\gamma)} - \sum_{n=0}^{\infty} \frac{\mathcal{B}_n}{\gamma - \theta_n^+} \cdot \frac{K_+(\theta_n^+)}{K_+(\gamma)}, \end{aligned} \tag{13}$$

where K represents the new Wiener-Hopf kernel defined by

$$K(\gamma) = j(\gamma) + \frac{1}{4\pi}\left(\mu_0 - i\mu_1\gamma - \mu_2\gamma^2\right).$$

All new variables in (13) are defined in an analogous way to [6]. We note that the solution is similar to Glegg's solution for the rigid cascade [6], except the original Wiener-Hopf kernel j is now replaced with the modified kernel K. We note that the poles of K are the same as j, whereas the zeros of K are generally distinct from those of j. The original kernel j is recovered when the boundaries are rigid and $\mu_1 = \mu_2 = \mu_3 = 0$.

We now invert the Fourier transform in the x-direction to obtain an expression for the acoustic field that is uniformly valid throughout the entire domain. The inversion is achieved by splitting the physical domain in a similar manner to [13]. Contour integration yields the final expression for the acoustic potential function, which takes a similar form to that of [13], and yields analytic expressions for the velocity and pressure fields.

4 RESULTS

In order to enable comparison against previous research, we consider the test cases analysed by Glegg [6] and Posson *et al.* [13] as outlined in table 2.

During the solution to the Wiener-Hopf problem associated with the scattering by the blade row with complex boundaries, we observed that the major difference with the rigid case is that the duct modes are modified. Consequently, we expect that modifications to the boundary conditions will have a strong impact on the pressure gradient across each blade, leading to significant effects on the unsteady loading. This hypothesis is tested in figure 2, where the unsteady loading for a rigid cascade is compared against the loading for a range of μ_0 values. The effects of complex boundaries on the unsteady loading are most significant at low frequencies ($\omega < 8$) and are far less significant at higher frequencies ($\omega > 8$) after the second acoustic mode is cut-on. Moreover, we observe a shift in the cut-on frequency of the passage modes, as reflected by the shifts in the peaks near $\omega \approx 12$.

Analytic expressions for the sound power output are available by a similar method to Glegg [6], and the downstream power output is plotted in figure 3. As anticipated by the Wiener-Hopf analysis, the cut-on frequencies of the acoustic modes are unchanged by the effects of complex boundaries. We note that there is a large jump in the sound power output for the case $\mu_0 = -5$ just above the cut-on frequency of the first mode. However, beyond this frequency, the sound power output monotonically decreases as μ_0 decreases. Similarly for the second and third modes, the sound power generated by blades with a small amount of compliance is generally much smaller than that for rigid blades $\mu_0 = 0$.

Varying μ_1 has the effect of increasing or decreasing the upstream and downstream pressure fields, as illustrated in figure 4. For example in figures 4c and 4d there is an increase in the downstream and upstream pressure fields respectively, whereas in figure 4b both the upstream and downstream pressure fields are reduced.

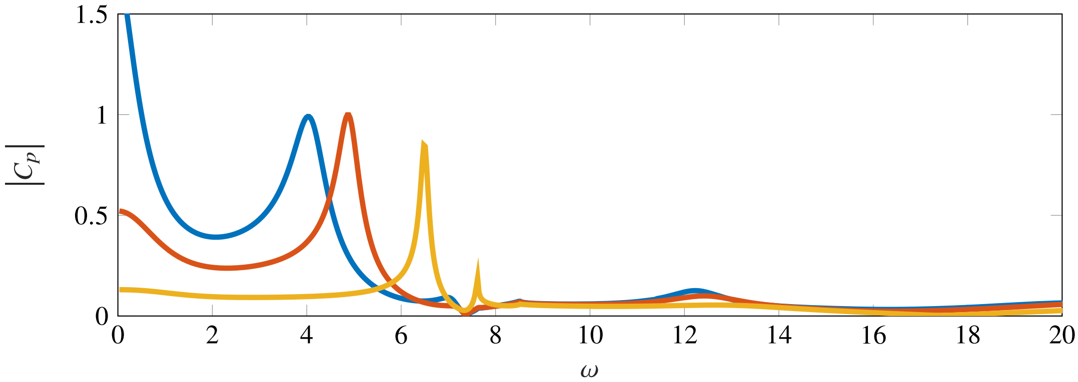

Figure 2. Unsteady lift for a range of frequencies. The aerodynamic and aeroacoustic parameters are defined in table 2 and correspond to those in figure 3 of [6]. In all plots $\mu_1 = \mu_2 = 0$ and the colours correspond to $\mu_0 = 0$, $\mu_0 = -1$ and $\mu_0 = -5$.

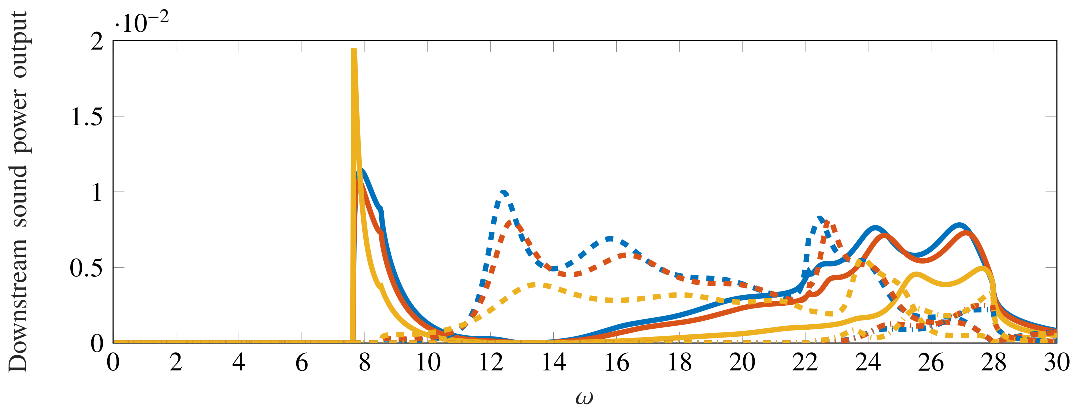

Figure 3. Downstream sound power output for a range of frequencies. The aerodynamic and aeroacoustic parameters are defined in table 2 and correspond to those in figure 9 of [6]. In all plots $\mu_1 = \mu_2 = 0$ and the colours correspond to $\mu_0 = 0$, $\mu_0 = -1$ and $\mu_0 = -5$. The line styles correspond to the first (———), second (- - - -), and third (-·-·-) modes respectively.

These effects can be attributed to the absorption and production of energy via the impedance condition applied to the blades.

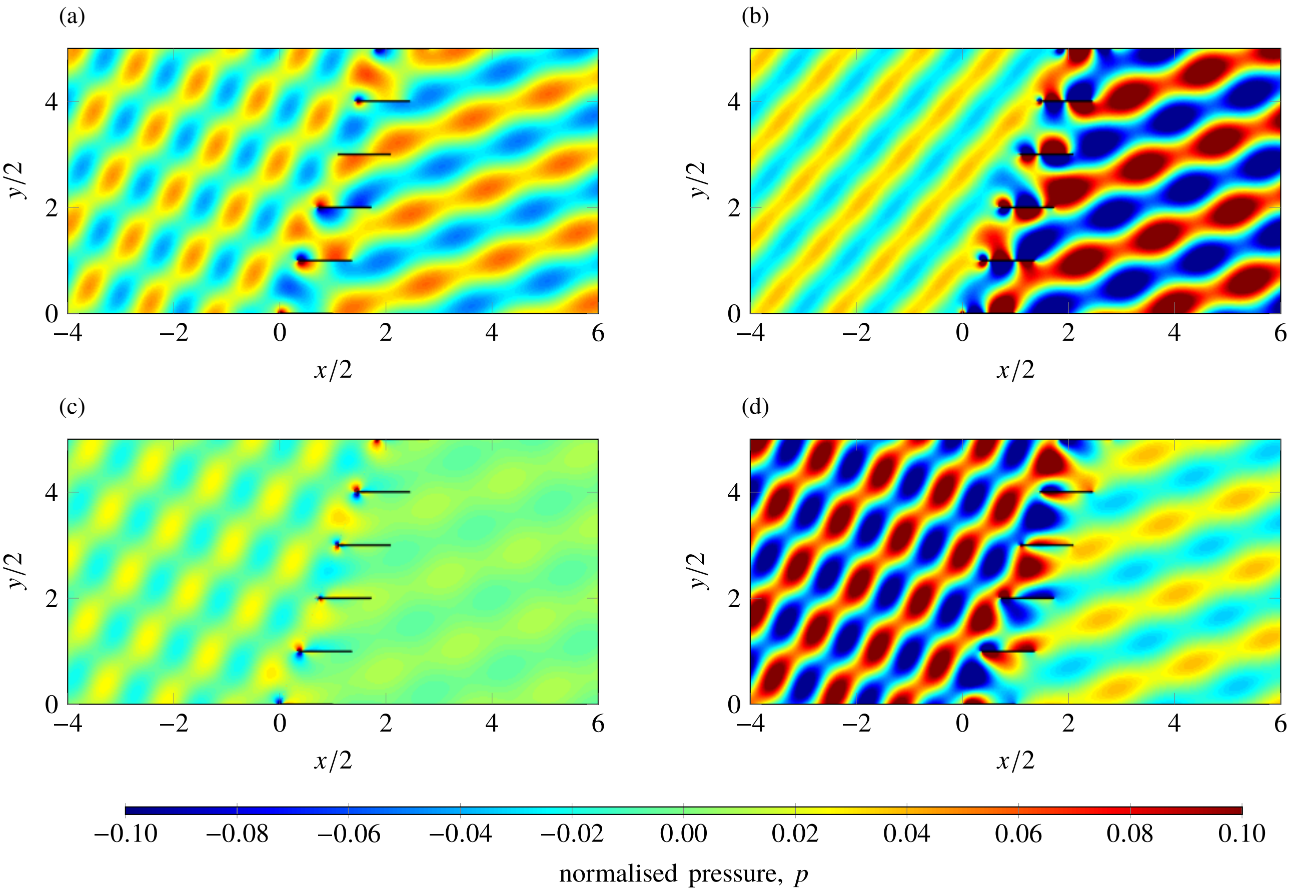

Figure 4. Scattered pressure for the parameters defined in Posson et *al.* [13] and reiterated in table 2. The boundary data is defined with $\mu_0 = \mu_2 = 0$ and (a) $\mu_1 = 0$ (i.e. rigid), (b) $\mu_1 = -2/3\mathrm{i}$, (c) $\mu_1 - 4/3\mathrm{i}$, (d) $\mu_1 = 4/3\mathrm{i}$.

5 CONCLUSIONS

We have derived an analytic solution for the scattering of an unsteady perturbation incident on a rectilinear cascade of flat plates with complex boundaries. The analytic nature of the solution means that it is extremely rapid to compute, and offers physical insight into the role played by different boundary conditions. In contrast with previous studies that focussed on the effects of rigid plates [6, 13], the formulation of the present research allows a range of boundary conditions to be studied with minimal effort, such as porosity, compliance and flow impedance. In terms of the spectral plane, the effect of modifying these boundary conditions is to change the locations of the zeros of the Wiener-Hopf kernel, whilst the poles are unchanged. Accordingly, the modal structure of the far-field scattered pressure is invariant under modifications to the flat plates' boundary conditions. Conversely, the modal structure of the near-field region undergoes large deformations since the zeros of the kernel correspond to the duct modes of the inter-blade region. This has a strong effect on the surface pressure fluctuations and unsteady loading, which has implications for the far-field sound. Future work will focus on more detailed analysis of elastic plates [1, 8] in order to explore the important effects of flutter and resonance.

Finally, this study shows that modified boundary conditions have a large impact on gust-cascade interaction noise and offers a useful design tool that can model aeroacoustic and aeroelastic effects.

ACKNOWLEDGEMENTS

P. J. B. acknowledges support from EPSRC grant 1625902 and L. J. A. acknowledges support from EPSRC grant EP/P015980/1.

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