Modal analysis of an enclosure acoustic space based on spectro-geometric method

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ABSTRACT

The Spectro-Geometric Method (SGM) has shown great efficiency in solving solid mechanics problems. In this investigation, acoustic eigenproblems of an enclosure acoustic space are solved using the SGM method. Under this solution framework, the sound pressure is invariantly expressed as a new trigonometric series in the form of the superposition of a Fourier cosine series supplemented by two sine terms. The use of the two sine terms is to ensure the series expansion converges uniformly and polynomially over the entire solution domain including the boundary surfaces. The Rayleigh–Ritz method is employed to derive the final characteristic equation for the acoustic system. The modal characteristics for an enclosure acoustic space can be directly obtained from solving a standard matrix eigenvalue problem. Several numerical examples have been presented to demonstrate the effectiveness and reliability of the SGM.

Keywords: Acoustics, Enclosure acoustic space, Spectro-Geometric Method (SGM) I-INCE Classification of Subjects Number(s): 21.6.9

1. INTRODUCTION

Acoustics is one of the most relevant topics that are very close to our daily life, and almost all the acoustics problems are related to enclosure acoustic spaces. The enclosed cavity is widely applied in many industries, such as the cab of the automobile, the cabin of the ship and that of the airplane. An acoustic modal analysis of such model can provide useful insights to physical understanding and effective guidance for acoustic design or noise control of enclosure acoustic space. Therefore, the research on the acoustic eigenproblems of an enclosure acoustic space is of great importance.

Over the past few decades, different methods have been employed by researchers to investigate the acoustical performance of enclosed cavity. Morse and Bolt (1) showed that the characteristics of the sound field in a room can be understood in terms of normal modes and the associated decay constant of each of these modes. They also developed a nonlinear transcendental characteristic equation through combining the assumed sound pressure modes with complex impedance boundary conditions on the walls. Maa (2) presented an exact solution for the behavior of sound in a rectangular room whose walls are covered non - uniformly with acoustical materials. In his study, experiments were also carried out to confirm the theoretical results. Roumeliotis et al (3) derived the analytical expressions for the eigenfrequencies of an acoustic cylindrical/rectangular cavity with an accentric inner small sphere. Fourier series method (4) and Chebyshev-Lagrangian method (5) were also utilized for acoustic analysis of rectangular cavity with general impedance walls.

From the review of these literatures, finite element method (FEM) (6, 7) and boundary element method (BEM) (8) are the major methods of solving acoustic problems. Although the numerical method is a powerful tool to model the acoustic system, the analytical method is usually preferred

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because of its convenience in the mechanism analysis and parametric study. In particular, modifying parameters in the analytical model is as simple as modifying the material properties and does not modify the solution procedures. In addition, it is well known that the reliability and accuracy of numerical method strongly depends on the size of the mesh. For high frequency range, a higher mesh density is required the large computational burden and this may restrict the element based methods to the low frequency range. Therefore, it is necessary and of great significance to develop a unified, efficient and analytical formulation which is capable of universally dealing with acoustic problems.

In the present work, a simple but efficient SGM is utilized to deal with the acoustic problem of an enclosure acoustic space. It has to be emphasized firstly that although the Lagrangian method is used both in Ref. (4) and in the present paper, the change of the base function is itself a fundamental point in the Lagrangian method. Comparing with the method used in Ref. (4), the SGM presented in this paper has its own advantage and this will be demonstrated in the next part of the paper. In the present work, the sound field is modeled with SGM and the impedance boundary is described by the work term in the Lagrangian function. The SGM solution is obtained and numerical cases are presented to demonstrate the effectiveness and accuracy of the present method.

2. THEORETICAL FORMULATIONS

2.1 Description of the acoustic cavity

With reference to Figure 1, the finite-sized annular segment cavity is assumed to have general impedance boundary conditions, occupying the spatial region ($a<r<b$, $0<\theta<\phi$, $0<z<h$) in the orthogonal cylindrical coordinates ($Or\theta z$). a and b are the inner and outer radius of annular segment cavity, respectively. h is the cavity length; ϕ is the sector angle of the cavity. The local orthogonal cylindrical coordinates ($O's\theta z$) is also shown in this figure, which will used in the following analysis. This kind of arbitrary boundary condition is described by assigning the boundary surfaces with corresponding impedance such that the classical Neumann or Dirichlet acoustic boundary condition can also be described with infinite or zero surface impedance, respectively. The impedance boundary condition can be expressed as:

$$\frac{\partial p}{\partial n} = -j\frac{\rho c}{Z_i}kp \tag{1}$$

where $j=\sqrt{-1}$, p is the sound pressure, n denotes the outgoing normal of the surface, ρ and c are the mass density of and the sound speed in the acoustic medium, respectively, k (=ω/c) is the wavenumber with ω being angular frequency, and Z_i denotes the acoustic impedance on the ith surface.

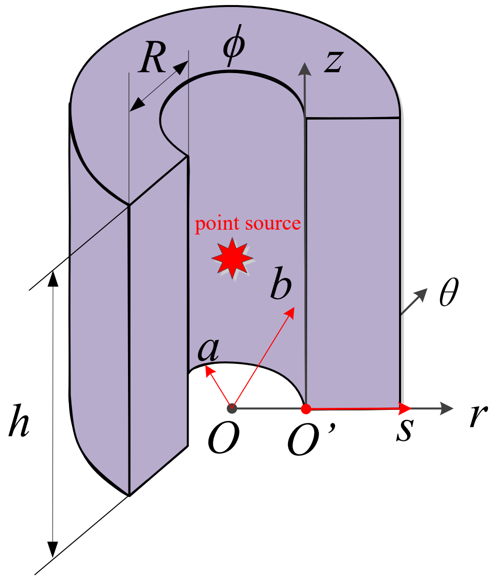

Figure 1 – This is a caption for the figure

Some references in room acoustics have indicated that impedance is a more reasonable variable than the absorption coefficient in characterizing the acoustic property of a wall surface, and the impedance value of specified commercial absorbing material can be obtained experimentally in practice. The impedance is defined as the complex ratio between the sound pressure at the surface and the air velocity normal to the surface just outside the surface. The real and imaginary parts of the complex impedance play distinctively different roles in determining the acoustic characteristics of a cavity. For a non-dissipative wall surface, the acoustic impedance is depicted by a pure imaginary number. The classical pressure release (or perfectly rigid) boundary conditions represent the special case when the imaginary part is practically equal to zero (or infinite).

2.2 Series representation of sound pressure

As is well-known, the sound pressure for rigid-walled acoustic cavity can be generally expressed as a 3-D Fourier cosine series. However, such a series expansion will become problematic when an impedance boundary condition is specified on one or more of the walls. It can be seen from Eq. (1) that the left side of the equation is identically equal to zero regardless of the actual value of the right side. This problem is mathematically related to the discontinuity to the first order derivative of sound pressure on the boundary surface under general boundary conditions. To eliminate this discontinuity, a new and simple three-dimensional version of spectro-geometric method previous developed for vibrations of plates and shells (9-11) will be utilized to describe the sound pressure inside the enclosure acoustic space.

$$p(s,\theta,z)=\sum_{m=-2}^{\infty}\sum_{n=-2}^{\infty}\sum_{l=-2}^{\infty}A_{mnl}\varphi_m(s)\varphi_n(\theta)\varphi_l(z) \qquad r=s+a \tag{2}$$

where A_{mnl} denotes the expansion coefficients, and

$$\begin{aligned}
\varphi_m(s)&=\begin{cases}\cos\lambda_m s & m\geq 0\\ \sin\lambda_m s & m<0\end{cases} \qquad \lambda_m=m\pi/R\\
\varphi_n(\theta)&=\begin{cases}\cos\lambda_n \theta & n\geq 0\\ \sin\lambda_n \theta & n<0\end{cases} \qquad \lambda_n=n\pi/\phi\\
\varphi_l(z)&=\begin{cases}\cos\lambda_l z & l\geq 0\\ \sin\lambda_l z & l<0\end{cases} \qquad \lambda_l=l\pi/h
\end{aligned} \tag{3}$$

The sine terms in the admissible function are introduced to overcome the potential discontinuities, along the walls of the cavity, of the sound pressure when it is periodically extended and sought in the form of trigonometric series expansion. As a result, the Gibbs effect can be eliminated and the convergence of the series expansion can be substantially improved. Instead of polynomial, the use of Fourier Sine Series can also ensure the smoothness of a displacement function and each of its derivatives along each wall of the acoustic cavity. This expression can also simplify the calculation process. It is very feasible to the analysis of complicate problems. In addition, the authors chose cosine series as the basic admissible functions of the acoustic cavity without containing the sine series because compared with sine series, the cosine series can provide more convergence speed and better accuracy when the acoustic cavity is with general impedance boundary conditions. More information about the difference between the cosine series and sine series can be seen in Ref (12).

2.3 The solution procedure

To obtain the solution of the acoustic problem, the Rayleigh-Ritz method is used due to the reliability of the results and convenience in modeling and solution process. The admissible sound pressure function is constructed sufficiently smooth in this study. Therefore, the weak solution obtained with Rayleigh-Ritz method is mathematically equivalent to the strong solution, which is solved by letting the series solution simultaneously satisfy both the governing equation (Helmholtz equation) inside the cavity and the boundary conditions on the cavity walls on a point-wise basis.

The corresponding Lagrangian for the annular segment cavity with arbitrary impedance boundary conditions can be expressed as

$$L=V-T-W_{\mathrm{ext}} \tag{4}$$

where V denotes the total acoustic potential energy stored in the enclosure space, T represents the total kinetic energy, and W_{ext} represents all the work done by the applied sources, which include the energy dissipation on the wall surfaces in the current case.

Denoting the mass density and the acoustic speed in the medium with ρ_0 and c_0, respectively, the particular expressions of the terms in the right side of Eq. (4) are given as follows:

$$V = \frac{1}{2\rho_0 c_0^2}\int_V p^2(s,\theta,z)dV = \frac{1}{2\rho_0 c_0^2}\int_0^R\int_0^\phi\int_0^h p^2(s,\theta,z)(s+a)dsd\theta dz \tag{5}$$

$$\begin{aligned} T &= \frac{1}{2\rho_0\omega^2}\int_V (\mathrm{grad}p)^2 dV \\ &= \frac{1}{2\rho_0\omega^2}\int_V\left[\left(\frac{\partial p}{\partial s}\right)^2 + \left(\frac{\partial p}{(s+a)\partial\theta}\right)^2 + \left(\frac{\partial p}{\partial z}\right)^2\right]dV \\ &= \frac{1}{2\rho_0\omega^2}\int_0^R\int_0^\phi\int_0^h\left[\left(\frac{\partial p}{\partial s}\right)^2 + \left(\frac{\partial p}{(s+a)\partial\theta}\right)^2 + \left(\frac{\partial p}{\partial z}\right)^2\right](s+a)dzd\theta ds \end{aligned} \tag{6}$$

where grad p means the gradient of sound pressure; ω is the frequency in radian.

The work done by the sound source inside cavity W_s can be described as

$$W_s = -\frac{1}{2}\int_V \frac{pQ}{j\omega}dV \tag{7}$$

where Q is the distribution function of a sound source. The familiar point source located at (x_0, y_0, z_0) inside the cavity can be expressed as

$$Q = Q_0\delta(x-x_0)\delta(y-y_0)\delta(z-z_0) \tag{8}$$

In Eq. (8), Q_0 is the volume velocity amplitude of the sound source, δ denotes the Kronecher delta function.

The work done by the boundary impedance at the point a on the surface equals to the pressure multiplied by the displacement, that is

$$\tilde{V}_a \quad p_a(u_a / j\omega) \tag{9}$$

in which, p_a is the sound pressure at point a and u_a is the velocity of particle a. Thus, the total work done by the impedance boundary which implies the mutual interactions between the impedance walls and the acoustic field in spatial space can be obtained by integrating Eq. (9) over all the impedance surface domain, namely,

$$W_B = \sum_{i=1}^{6}\int_{s_i}\tilde{V}_a \quad \frac{1}{2}\sum_{i=1}^{6}\int_{s_i}\frac{p^2}{j\omega Z_i}ds_i \tag{10}$$

Therefore, the W_{ext} can be represented as

$$W_{\mathrm{ext}} = W_{\mathrm{B}} + W_{\mathrm{s}} \tag{11}$$

Let the derivatives of the Lagrangian with respect to the unknown coefficients of the sound pressure equal zero, that is,

$$\frac{\partial L}{\partial A_{mnl}} = 0 \tag{12}$$

Thus, one is able to obtain the characteristic equation of the acoustic system, in a matrix form, as

$$\left(\boldsymbol{K} + \omega\boldsymbol{Z} + \omega^2\boldsymbol{M}\right)\boldsymbol{X} = \boldsymbol{Q} \tag{13}$$

where **K** and **M** are the stiffness and mass matrix of the acoustic cavity, respectively; **Z** is the damping matrix due to the dissipative effect of the impedance boundary conditions over the cavity walls, and **Q** is the external load vector related to the sound source. **X** is the series expansion coefficient vector of the sound pressure expression. Once there is no sound source inside the cavity, the elements on the right side of the characteristic equation equal zero. In this case, Eq. (13) represents the free modal characteristic equation of the acoustic cavity.

It should be noted that the characteristic equation involves the first-order and second-order terms of oscillation frequency. Eq. (13) can be rewritten in state space form

$$\left(\boldsymbol{R} - \omega\boldsymbol{S}\right)\boldsymbol{E} = \boldsymbol{F} \tag{14}$$

where

$$\boldsymbol{R}=\begin{bmatrix} \boldsymbol{0} & -\boldsymbol{K} \\ -\boldsymbol{K} & -\boldsymbol{Z} \end{bmatrix},\ \boldsymbol{S}=\begin{bmatrix} -\boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M} \end{bmatrix},\ \boldsymbol{E}=\begin{bmatrix} \boldsymbol{A} \\ \omega\boldsymbol{A} \end{bmatrix} \text{ and } \boldsymbol{F}=\begin{bmatrix} \boldsymbol{0} \\ -\boldsymbol{Q} \end{bmatrix} \tag{15}$$

To determine the free modal characteristic of the acoustic cavity, the external load vector **F** on the right side of Eq. (14) should be set to zero. It is evident now that the eigenvalues and eigenvectors can be easily determined from solving a standard matrix eigenvalue problem. For a given natural frequency, the corresponding eigenvector actually contains the series expansion coefficients which can be used to construct the physical mode shape based on Eq. (2).

3. NUMERICAL RESULTS AND DISCUSSIONS

To demonstrate the accuracy and usefulness of the presented technique, the free modal analysis results for acoustic cavity are determined with the present approach and compared with theoretical solutions or results obtained with finite element method (FEM).

As mentioned in the theoretical section, the sound pressure is expressed as an improved triangular series expansion, which includes infinite terms. However, the series expansion must be truncated to *M*, *N* and *L* in actual calculations to obtain the reliable results with acceptable accuracy due to the limited speed, the capacity and the numerical accuracy of computers. Let us consider an annular segment cavity with the following parameters: inner radius (a), outer radius (b), sector angle (ϕ) and height (h) are 0.5m, 1m, π/2 and 2m. The density (ρ_0) and the acoustic speed (c_0) for the acoustic medium are 1.21kg/m^3 and 340m/s, respectively. When the acoustic cavity walls are rigid, the particle at the surface of these walls is zero. It also should be noted that the impedance of the walls does not work on the acoustic field. Therefore, all the elements in the impedance matrix **Z** in Eq. (13) are zero. Thus, there are two different ways to deal with the rigid boundary. One is directly delete the impedance matrix and only retain the stiffness and mass matrixes in the characteristic equation. Another one is setting the rigid walls with infinite impedance (Z_i). In the actual numerical calculations, the infinite impedance for a rigid wall can be treated as j10^{10} or larger.

For a rigid annular segment cavity, the theoretical solution of eigenvalues can be calculated as following (13)

$$f_{ijk}=\frac{c_0}{2\pi}\sqrt{\frac{\lambda_{jk}^2}{b^2}+\frac{i^2\pi^2}{h^2}} \quad (i,k=0,1,2,3....) \tag{16}$$

where

$$j=\frac{n\pi}{\phi} \quad (n=0,1,2,3....) \tag{17}$$

j will be integer only if ϕ is a submultiple of $n\pi$. λ_{jk} can be found in Table 1.

Table 1 – Tabulated values for λ_{jk}

a/b	λ_{jk}							
	j							
	k	0	1	2	3	4	5	6
0.3	0	0	1.5821	2.9685	4.1801	5.3130	6.4147	7.5011
	1	4.7058	5.1374	6.2738	7.7213	9.1526	10.475	11.721
0.5	0	0	1.3547	2.6812	3.9577	5.1752	6.3389	7.4622
	1	6.3932	6.5649	7.0626	7.8401	8.8364	9.9858	11.228

The first six natural frequencies are presented in Table 2. As a comparison, the theoretical solutions are also given in the table. In this table, the truncated terms in both s, θ and z directions *M*, *N* and *L* are altered simultaneously from 3 to 8. It can be observed that very good agreement of the results is obtained even though the truncated number is 3. To check the convergence speed of the present method, a further investigation of the natural frequencies is shown in Figure 2. It can be seen that the first 30 natural frequencies can be considered as stable when the truncated number is increased to 5. From the results, it will be known that the present method shows excellent convergence speed and this will be helpful to expand the analysis into the higher frequency range.

Table 2 – First six natural frequencies for a rigid annular segment cavity

Mode number	1	2	3	4	5	6
M=N=L=3	85.000	145.09	168.15	170.00	223.49	255.00
M=N=L=4	85.000	145.09	168.15	170.00	223.49	255.00
M=N=L=5	85.000	145.09	168.15	170.00	223.49	255.00
M=N=L=6	85.000	145.09	168.15	170.00	223.49	255.00
M=N=L=7	85.000	145.09	168.15	170.00	223.49	255.00
M=N=L=8	85.000	145.09	168.15	170.00	223.49	255.00
Theoretical	85.000	145.09	168.15	170.00	223.49	255.00

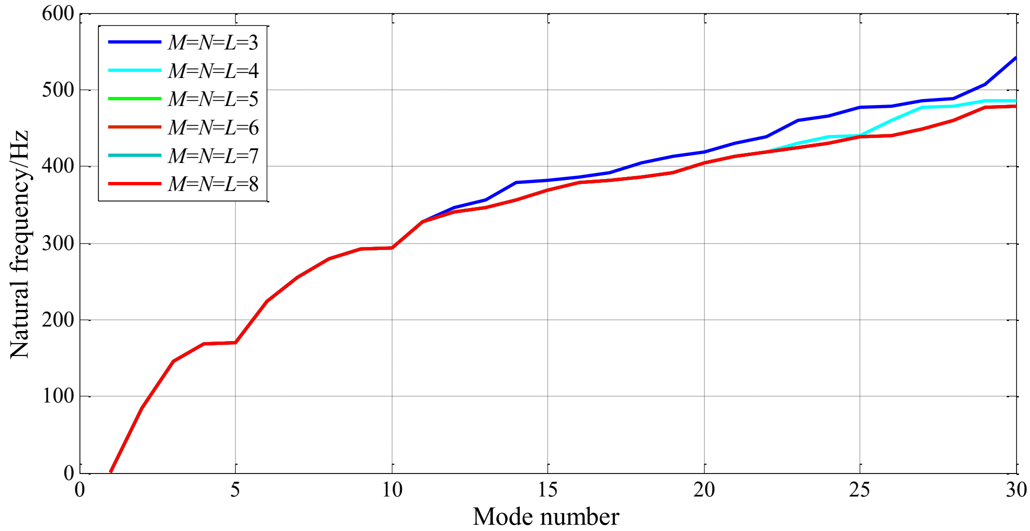

Figure 1 – Convergence of the present method

The meshless and parametric modeling makes it better suitable for parametric study. Now the numerical model developed in this investigation is utilized to study the effects of sector angle on modal characteristics of acoustic cavity. Table 3 listed the first six natural frequencies for annular segment cavity with different sector angles. The parameters are: a=0.3m, b=1m and h=2m.

Table 3 – First six natural frequencies for a rigid annular segment cavity with different sector angle

sector angle ϕ	1	2	3	4	5	6
$\pi/3$	0	85.000	170.00	226.19	241.64	254.64
theoretical	0	85.000	170.00	226.19	241.64	254.64
$\pi/2$	0	85.000	160.63	170.00	181.74	233.89
theoretical	0	85.000	160.63	170.00	181.74	233.89
π	0	85.000	85.610	120.64	160.63	170.00
theoretical	0	85.000	85.666	120.68	160.63	170.00
$3\pi/2$	0	57.937	85.000	102.87	111.97	140.58

It can be seen from Table 3 that the sector angle plays a very important role on the modal characteristic of acoustic cavity. When the sector angle $\phi\leq\pi$, the second natural frequency will not change with the increase of sector angle ϕ.

4. CONCLUSIONS

A meshless and parametric modeling method, the so-called Spectro-Geometric Method (SGM), is extended to solve acoustic eigenproblems of an enclosure acoustic space. Under the current framework, the admissible function of the sound pressure, regardless of boundary conditions, is expressed in spectral form, as a new and simple of trigonometric series expansion with an accelerated polynomial rate of convergence. The influence of boundary impedance on the cavity is considered through the work done by the impedance surfaces. The unknown series expansion coefficients of the sound pressure are determined by using the Rayleigh-Ritz method. The accuracy and reliability of the SGM prediction have been demonstrated numerically. Unlike most existing techniques, the current method offers a unified solution for annular segment cavity with various impedance boundary conditions, and the modification of impedance boundary condition is as simple as changing the material and geometric parameters. Although this study is focused on free modal analysis, the forced response of the acoustic system can be readily considered by accordingly modifying work done by the sound source.

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