A Meshless Method for the Acoustic Wave Propagation with the Weighted Least Squares Filtering

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ABSTRACT
Meshless methods based on the collocation of hyperbolic equations are commonly stabilized using upwind approximation of the derivatives. This approach leads to stabilization of the numerical scheme, but its accuracy is significantly reduced by dissipation. For linear systems, however, high order of accuracy can be achieved by using central-type schemes and the stability of the numerical method can be ensured via filtering of the numerical solution. We present a simulation of the acoustic wave propagation phenomenon using meshless method that solves the Linearized Euler equations with the Weighted least squares filtering.

Keywords: Meshless methods, Acoustic wave propagation, Weighted least squares

1. INTRODUCTION

This paper is devoted to study the sound propagation phenomenon using a numerical simulations of Linearized Euler equations (LEE), cf. [1], [2], with a meshless method adjusted for linear systems. This numerical technique will be investigated on benchmark problems according to stability, robustness and accuracy of the method that are crucial properties in aeroacoustic applications.

Firstly, basic concepts and the derivation of the meshless method, cf. [3], for linear hyperbolic systems with weighted least squares (WLSQ) filtering is introduced, cf. [4], [5], [6]. The WLSQ filtration technique serves primarily for the stabilization of the numerical scheme and can be utilized under the assumption that the governing equations are linear and the solution is smooth enough. In contrast to the stabilization that is enforced by upwind approximation of derivatives, cf. [7], [8], [9], [10], [11], [12], [13], this approach does not burden the solution with dispersion and dissipation errors allowing to achieve high order of the method without any demanding accuracy improvements in terms of computational speed and memory requirements.

Simple benchmark problems for the acoustic wave propagation, cf. [14], [15], [16], are then solved and the numerical solution is compared with known analytical solutions allowing to perform the error estimation. A Perfectly matched layer (PML), cf. [17], [18], [19], [20], [21], is used for the implementation of the non-reflecting boundary condition. After the collocation of governing equations, accurate temporal discretization is achieved by low-dissipation and low-dispersion runge-kutta scheme, cf. [22].

2. MESHLESS METHOD WITH WLSQ FILTERING

2.1 Basic notation

Definition 2.1 Let $\Omega \subset \mathbb{R}^d$ be the domain and $\Gamma$ its boundary. The global cloud $\hat{\Omega}$ is defined as a finite set of points from $\overline{\Omega}$ which discretizes the closed domain $\overline{\Omega}$. We write

$$\hat{\Omega} = \{x_i\}_{i=1}^n.$$  (1)

The $i$-th local cloud $\hat{\Omega}_i$ is defined as a finite set

$$\hat{\Omega}_i = \hat{\Omega} \cap \Omega_i.$$  (2)

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where \( \Omega_i = \{ x \in \mathbb{R}^d, \| x - x_i \|_p < r_i \} \) is an open ball in \( p \)-norm and the union of \( \Omega_i \) creates the covering of \( \Omega \), i.e. \( \Omega \subseteq \bigcup_{i=1}^n \Omega_i \). Similarly we write

\[
\hat{\Omega}_i = \{ x^i_j \}_{j=1}^{n_i},
\]

where \( n_i \) is the number of points in the local cloud \( \hat{\Omega}_i \). Moreover, the particular point \( x_i = x^i_1 \) is called the star point of the local cloud \( \hat{\Omega}_i \).

Discretization of a domain by a particular global cloud and two selected local clouds in \( p = 2 \) and \( p = \infty \) norm with their star points is illustrated in Fig. 1.

![Figure 1: Domain \( \Omega \) with boundary \( \Gamma \), two domains \( \Omega_i \) (open balls) and their star points \( x^i_1 \).](image)

Given a domain of interest \( \bar{\Omega} \), the system of local clouds \( \hat{\Omega}_i, i = 1, \ldots, n \) is calculated in the preprocessing step of the simulation. Practically, efficient algorithms for nearest neighbor search such as the k-d tree are used. Therefore, the number of points \( n_i \) is controlled by the radius of an open ball \( r_i \) or it can be manually prescribed.

### 2.2 Weighted Least Squares Method

For every local cloud \( \hat{\Omega}_i \) we wish to find a local approximation \( \hat{w} : \Omega_i \rightarrow \mathbb{R} \) in the form

\[
\hat{w}(x) = \sum_{l=1}^{m} \alpha_l p_l(x) = p^T(x) \alpha
\]

where

\[
\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)^T \in \mathbb{R}^m
\]

\[
p^T(x) = (p_1(x), p_2(x), \ldots, p_m(x))
\]

Functions \( p_l : \mathbb{R}^d \rightarrow \mathbb{R}, \ l = 1, \ldots, m \), form a basis \( \mathcal{B} \) of an approximation space \( \mathcal{F} = \text{span}(\mathcal{B}) \). The complete (multivariate) polynomial basis of degree \( \nu \), cf. [3], [8] is adopted in this work.

Given a degree of polynomial \( \nu \), for the number of basis functions \( m \) holds

\[
m = \frac{1}{2} (\nu + 1)(\nu + 2).
\]

For \( d = 2 \) and \( \nu = 3 \), e.g.

\[
\mathcal{B} = \{ 1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3 \}.
\]
\textbf{2.3 WLSQ approximation}

Standard WLSQ method finds the coefficients $\alpha$ in (4) by minimizing the objective function $J(\alpha)$ with weights

$$J(\alpha) = (P\alpha - w)^T \Phi (P\alpha - w),$$

where

$$P := \begin{pmatrix} p^T(x_1^i) \\ p^T(x_2^i) \\ \vdots \\ p^T(x_n^i) \end{pmatrix}, \quad \Phi := \text{diag}\left\{ \phi(x_j^i) \right\}_{j=1}^{n_i}, \quad w = (w_1, \ldots, w_{n_i})^T$$

denotes the moment matrix of type $(n_i \times m)$, diagonal matrix $\Phi$ of order $n_i$ and $w$ the vector of given function values, respectively. The weighting function $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ prescribes the contribution of every point to $J(\alpha)$ in the local cloud according the distance to the star point $x_1^i$, cf [8].

Solution to the system of normal equations

$$P^T\Phi P\alpha = P^T\Phi w,$$

then yields

$$\alpha = Cw,$$

where

$$C := (P^T\Phi P)^{-1}P^T\Phi$$

is matrix of type $(m \times n_i)$. We suppose that the number of points $n_i$ in $\Omega_i$ is greater than the number of basis functions $m$, i.e. $n_i > m$ holds.

\textbf{2.3.1 Weighting function}

Gaussian weighting function $\phi(x)$ given by 3 parameters $(\omega, k, \gamma) = (3, 2, 1.01)$, cf. [7], [8], is adopted in this work

$$\phi(x) = \frac{e^{-h^k} - e^{-\omega^k}}{1 - e^{-\omega^k}}, \quad x \in \Omega_i$$

where $h := \frac{d\omega}{d_{\max}^{-\gamma}}$, $d := \|x^i_1 - x\|$, $d_{\max} := \max\{\|x^i_j - x^j\| : j = 1, 2, \ldots, n_i\}$, $x^i_j \in \Omega_i$.

\textbf{2.4 Shape functions}

The function $\hat{w}(x)$ can be rewritten using linear combination in the form

$$\hat{w}(x) = \sum_{j=1}^{n_i} \psi_j(x)w_j = \psi^T(x)w,$$

where $\psi(x) = (\psi_1(x), \psi_2(x), \ldots, \psi_{n_i}(x))^T$ is the vector of shape functions, cf. [3]. The definition of $\psi(x)$ follows directly after the substitution of (12) into (4)

$$\psi(x) = p^T(x)C \quad \text{or} \quad \psi_j(x) = p^T(x)c_j,$$

where $c_j$ is the $j$-th column of matrix $C$.

Finally, in the multi–index notation $\alpha = (a_1, a_2, \ldots, a_d)$, the derivatives of $\hat{w}(x)$ using the shape functions can be easily obtained as

$$\partial^\alpha \hat{w}(x) = \partial^\alpha \psi^T(x)w = \left( \partial^\alpha p_1(x), \partial^\alpha p_2(x), \ldots, \partial^\alpha p_m(x) \right) Cw.$$
2.5 WLSQ interpolation

By modifying the WLSQ approximation \( \hat{w}(x) \) by the interpolation constraint

\[
\hat{w}(x_i^1) = w_1, \quad \text{(18)}
\]

the value \( w_1 \) is reproduced at the star point \( x_i^1 \) exactly. This interpolation property of WLSQ proved to be very useful in discretization of governing equations, see section 3. As a consequence the expression (4) becomes

\[
\hat{w}(x) = w_1 + \sum_{l=2}^{m} \alpha_l p_l(x), \quad p_l(x_1) = 0, \quad l = 2, \ldots, m. \quad \text{(19)}
\]

The coefficients \( \alpha = (\alpha_2, \ldots, \alpha_m) \) are then sought in the WLSQ sense leading to a modified system of normal equations in form

\[
P^T \Phi P \alpha = P^T \Phi w. \quad \text{(20)}
\]

where

\[
P := \begin{pmatrix} p_2(x_1^2) & p_3(x_1^2) & \ldots & p_m(x_1^2) \\ p_2(x_1^3) & p_3(x_1^3) & \ldots & p_m(x_1^3) \\ \vdots & \vdots & \ddots & \vdots \\ p_2(x_n^i) & p_3(x_n^i) & \ldots & p_m(x_n^i) \end{pmatrix}, \quad W := \begin{pmatrix} -1 & 1 & 0 & \ldots & 0 \\ -1 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & \ldots & 1 \end{pmatrix},
\]

and

\[
\Phi := \text{diag} \{ \phi(x_j) \}_{j=2}. \quad \text{(22)}
\]

Similarly, if we denote

\[
\tilde{C} := (P^T \Phi P)^{-1} P^T \Phi W \quad \text{(23)}
\]

then the expression (12) becomes

\[
\alpha = \begin{pmatrix} e^T \\ \tilde{C} \end{pmatrix} w, \quad \text{where} \quad e^T = (1, 0, \ldots, 0). \quad \text{(24)}
\]

Thus, the interpolation constraint (18) can be gently implemented in the WLSQ approximation context as described in section 2.3.

3. MESHLESS METHOD FOR A LINEAR HYPERBOLIC SYSTEM

3.1 2D Linearized Euler equations

Let us consider hyperbolic non-homogeneous 2D Linearized Euler equations (LEE) as a model for the wave propagation phenomenon in the following matrix form

\[
\frac{\partial w'}{\partial t} + \mathbb{A}_1(w_0) \frac{\partial w'}{\partial x} + \mathbb{A}_2(w_0) \frac{\partial w'}{\partial y} = S, \quad x = (x, y) \in \Omega, \quad t > 0. \quad \text{(25)}
\]

where \( w'(x, t) = (\rho'(x, t), u'(x, t), v'(x, t), p'(x, t))^T \) denotes the time dependent acoustic variables (density, velocity components and pressure) and \( \mathbf{w}_0(x) = (\rho_0(x), u_0(x), v_0(x), p_0(x))^T \) steady mean flow variables corresponding to the underlying flow field. Jacobian matrices of the system (25) are given as

\[
\mathbb{A}_1(w_0) = \begin{pmatrix} u_0 & \rho_0 & 0 & 0 \\ 0 & u_0 & 0 & \frac{1}{\rho_0} \\ 0 & 0 & u_0 & 0 \\ 0 & \gamma p_0 & 0 & u_0 \end{pmatrix}, \quad \mathbb{A}_2(w_0) = \begin{pmatrix} v_0 & 0 & \rho_0 & 0 \\ 0 & v_0 & 0 & 0 \\ 0 & 0 & v_0 & \frac{1}{\rho_0} \\ 0 & 0 & \gamma p_0 & v_0 \end{pmatrix}, \quad \text{(26)}
\]

where \( \gamma = 1.4 \) is the ratio of specific heats and \( S = S(x, t) \) represents the acoustic source term, cf. [1], [2].

The initial–value boundary problem (IVBP) for LEE then consists of the equation system (25) with the initial and boundary condition

\[
\begin{align}
\text{(IC)} \quad & w'(x, 0) = w'_\text{in}(x), \quad x \in \Omega, \quad \text{(27)} \\
\text{(BC)} \quad & w'(x, t) = w'_\Gamma(x, t), \quad x \in \Gamma = \partial \Omega, \quad t > 0. \quad \text{(28)}
\end{align}
\]
3.2 Spatial discretization

Let the global cloud \( \hat{\Omega} = \{ x_i \}_{i=1}^n \) as well as the local clouds structure, i.e. for each point \( x_i \in \hat{\Omega} \) the corresponding local cloud \( \hat{\Omega}_i \), consisting of \( n_i \) points is known. Local WLSQ approximation \( \hat{w}_i(x, t) \) of the acoustic variables \( w'(x, t) \) with the interpolation condition is then constructed at domains \( \Omega_i \) as follows

\[
\hat{w}_i(x, t) = \sum_{j=1}^{n_i} \psi_j^i(x) w_j(t), \quad i = 1, \ldots, n.
\]  

(29)

Similarly, the fluxes \( F_k(w') := \kappa_k(w_0)w' \), \( k = 1, 2 \) can be approximated locally at domains \( \Omega_i \) as follows

\[
\hat{F}_{k,i}(w') = \sum_{j=1}^{n_i} \psi_j^i(x) F_k(w_j(t)), \quad i = 1, \ldots, n.
\]  

(30)

In order to obtain the semi–discrete form of governing equation, the collocation method is adopted, i.e. the expression

\[
\left[ \frac{\partial \hat{w}_i(x, t)}{\partial t} + \frac{\partial \hat{F}_{1,i}(w')}{\partial x} + \frac{\partial \hat{F}_{2,i}(w')}{\partial y} - S(x, t) \right]_{x=x_i} = 0
\]  

(31)

has to be satisfied. The collocation then leads to a system of ordinary differential equations (ODE) in form

\[
\frac{dw_i(t)}{dt} = - \left( \sum_{j=1}^{n_i} \frac{\partial \psi_j^i(x_i)}{\partial x} F_1(w_j(t)) + \sum_{j=1}^{n_i} \frac{\partial \psi_j^i(x_i)}{\partial y} F_2(w_j(t)) \right) + S(x_i, t)
\]  

(32)

where \( w_i(t) \equiv \hat{w}_i(t), \ x_i \notin \Gamma \). From (28) we have \( w_j(t) = w_j(x_j, t) \) for \( x_j \in \Gamma \) and from (27) we obtain \( w_j(0) = w_m(x_j) \). The interpolation condition in the WLSQ approach allows to simplify the left side of the equation system (32) avoiding the necessity of solution of the system of \( n \) linear equations. Therefore, using WLSQ with interpolation condition greatly reduces the computation costs and improves stability, cf. [8], [9], [10]. The system of ODE (32) is then solved with high order low-dissipation and low-dispersion runge-kutta scheme optimized for wave propagation problems, cf. [22].

3.3 Filtering of the solution

The numerical scheme (32) derived in section 3.2 can be viewed as a generalized finite differences scheme based on central differences which is not stable in general, cf. [11], [12]. The stability of the meshless method is then usually enforced via upwind approximation of fluxes similarly to Finite volume methods, cf. [8], [10], [13], [15]. This technique allows to solve systems of nonlinear hyperbolic PDE but on the other hand increases complexity and overall computational costs of the method.

For linear problems, e.g. the acoustic wave propagation described by LEE, the stabilization procedure of the meshless method can be gently achieved in the same WLSQ context as for the discretization of governing equations. Generally, filters based on least squares minimization are known under the name Savitzky–Golay filters, cf. [4], [5], [6]. Not only the stability, but also robustness and accuracy of the meshless method is then affected by the WLSQ filter.

The \( i \)-th local cloud \( \hat{\Omega}_i = \{ x_i^j \}_{j=1}^{n_i} \) can be utilized not only for the discretization of governing equations, as described in section 3, but also for spatial filtering. In the latter case we use standard WLSQ approximation (see section 2.3), to obtain the vector of coefficients \( \alpha \). During the time integration (after every time step) the solution \( w_i(t) \approx w'(x_i, t) \) is replaced with the filtered value

\[
w_i(t) = p^T(x_i) \alpha
\]  

(33)

at the star point \( x_i \).

3.4 Boundary conditions

Two main types of boundary conditions are adopted in this work, namely the solid–wall and non-reflecting BC modeled using buffer techniques, especially the Perfectly matched layer. For an overview of these methods see [17], [18].
3.4.1 Solid–wall boundaries

Solid–wall slip boundary condition is prescribed at points where the acoustic waves should be reflected. The acoustic velocity vector \( \mathbf{v'} = (u', v') \) at the boundary point \( \mathbf{x} \in \Gamma \) is forced to stay orthogonal to the normal vector \( \mathbf{n} = (n_x, n_y) \), i.e.

\[
\mathbf{v'} \cdot \mathbf{n} = 0,
\]

which can be gently implemented by replacing the acoustic velocity vector \( \mathbf{v'} \) with its tangential component

\[
\mathbf{v'} - (\mathbf{v'} \cdot \mathbf{n}) \mathbf{n}.
\]

3.4.2 Non–reflecting boundaries

The non–reflecting BC is prescribed at domain boundaries, where waves should leave the computational domain without any reflection. It can be achieved artificially by designing a layer of points around the domain of interest that will affect the solution in such a way that the incoming waves will dissipate. The PML is implemented in our numerical experiments, cf. [19], [20], [21].

4. NUMERICAL EXPERIMENTS

4.1 2D Wall-reflected Acoustic Pulse Problem

This benchmark problem is designed to verify the implementation of the wall boundary condition, the accuracy and stability of a numerical method when a reflection of an acoustic wave from the solid wall occur. Let us consider a 2-dimensional domain \( \Omega = (-100, 100) \times (0, 150) \) that is depicted in Fig. 2 together with the initial pulse located at \( (x_p, y_p) = (0, 25) \). The wall boundary condition is prescribed at line \( y = 0 \) and the rest of the boundary is treated as an outflow BC. We suppose the underlying medium at rest, which implies that the mean flow velocity equals zero and therefore the Mach number \( M = 0 \) and for \( w_0 \) it holds

\[
\rho_0 = 1, \quad u_0 = v_0 = 0, \quad p_0 = \rho_0 \tfrac{5}{7}.
\]

The Initial-value boundary problem consists of the equation system (25) without the source term \( \mathbf{S} \), above mentioned boundary conditions and initialized by the initial condition \( \mathbf{w'}(x, y, 0) \) given as follows

\[
\mathbf{w'}_{\text{in}}(x, y) = \varepsilon \exp(-\kappa r_p^2) (1, 0, 1)^T,
\]

where the radius \( r_p = \sqrt{(x - x_p)^2 + (y - y_p)^2} \) and \( \kappa = (\ln 2)/b^2 \). The amplitude and the half–width of the acoustic initial pulse are determined by \( \varepsilon = 1 \) and \( b = 5 \), respectively.

Figure 2: Rectangular domain \( \Omega \), wall (solid line) and external boundary (dashed line).
4.1.1 Analytical solution

The analytical solution $p'_{ex}(x, y, t)$ to this benchmark problem, cf. [14], is given as

$$p'_{ex}(x, y, t) = \frac{1}{2\kappa} \int_0^\infty \exp\left(-\frac{\zeta^2}{4\kappa}\right) \cos(\xi t) [J_0(\xi \eta) + J_0(\xi \zeta)] \xi \, d\xi$$  \hspace{1cm} (38)

where $\eta = \sqrt{(x - Mt - x_p)^2 - (y - y_p)^2}$, $\zeta = \sqrt{(x - Mt - x_p)^2 + (y - y_p)^2}$ and $J_0$ is the Bessel functions of the first kind.

4.1.2 Error estimation

In order to estimate the convergence rate of the numerical solution, we present the error evaluation for various irregular point distributions in the domain $\Omega$. The analysis was performed with $\nu = 4568$, 8897, 17965 and 36006 points. Spatial discretizations of governing equations using the polynomial basis of degree $\nu = 2, \ldots, 6$, cf. section 2.2, was taken into account. The time step was adjusted according to the density of point distribution in $\Omega$, namely $dt = 0.125, 0.125, 0.05$ and $0.025$ were used, respectively.

Table 1 compares the estimation of the analytical and numerical solution for the acoustic pressure measured by 3 types of error, the maximal absolute error $E_{max}$, average error $E_{av}$ and $L^2$-error $E_{\parallel \parallel 2}$ defined as

$$E_{max} = \max |p'_{ex}(x_i, y_i, T) - p'(x_i, y_i, T)|,$$ \hspace{1cm} (39)

$$E_{av} = \frac{1}{N} \sum_{i=1}^{N} |p'_{ex}(x_i, y_i, T) - p'(x_i, y_i, T)|,$$ \hspace{1cm} (40)

$$E_{\parallel \parallel 2} = \left(\frac{1}{N} \sum_{i=1}^{N} (p'_{ex}(x_i, y_i, T) - p'(x_i, y_i, T))^2\right)^{\frac{1}{2}}.$$ \hspace{1cm} (41)

where $p'_{ex}(x_i, y_i, T)$ is the analytical solution (38) for the acoustic pressure at points along the line $x = 0$ and time $T = 75$. Values of the numerical acoustic pressure at points along the line $x = 0$ were linearly interpolated from the solution $p'(x, y, T)$ at irregularly distributed points in $\Omega$.

<table>
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<tr>
<th>$E_{max}$</th>
<th>$\nu = 2$</th>
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<th>$\nu = 4$</th>
<th>$\nu = 5$</th>
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Table 1: Maximal $E_{max}$, average $E_{av}$ and $L^2$-error $E_{\parallel \parallel 2}$ for various degrees of polynomial approximation.

4.1.3 Numerical solution

The comparison of the acoustic pressure $p'(x, y, T)$ along the line $x = 0$ and analytical solution at time $T = 75$ is shown in Fig. 3. Moreover, the acoustic pressure plotted over time at the point $x = (0, 0)$ is
depicted in Fig. 4. The time integration was performed with time step $dt = 0.025$ that corresponds to the unstructured point distribution with $n = 36006$ number of points.

Figure 3: Acoustic pressure $p'(x, y, T)$ along the line $x = 0$.

Figure 4: Acoustic pressure $p'(x, y, t)$ plotted over time at the point $x = (0, 0)$.

Figure 5: Acoustic pressure $p'(x, y, T)$ contours at time $T = 10, 25, 50, 75$. 
Let us consider a 2-dimensional domain $\Omega = (-100, 100)^2$ and the subsonic uniform mean flow in $x$-direction, i.e. we prescribe the vector of mean flow variables $w_0$ as follows

$$
\rho_0 = 1, \quad u_0 = 0.5, \quad v_0 = 0, \quad p_0 = \frac{\rho_0}{\gamma} = \frac{5}{7}.
$$

The speed of sound $a_0$ and the Mach number $M_0$ of the underlying flow are therefore

$$a_0 = \sqrt{\frac{\gamma p_0}{\rho_0}} = 1 \quad \text{and} \quad M_0 = 0.5.
$$

In this example, the acoustic source term on the right-hand side of the system (25) is the periodic perturbation in the acoustic density and pressure at the origin $x_s = y_s = 0$ which is given as

$$S(x, t) = e^{-\alpha ((x-x_s)^2 + (y-y_s)^2)} \sin(\omega t)(1, 0, 0, 1)^T
$$

where $\alpha = \ln(2)/2$, amplitude $\varepsilon = 0.5$ and angular frequency $\omega = 2\pi/30$.

The domain $\Omega$ is extended by the PML $\Omega_e$ where stretching of points and artificial dissipation is applied in order to suppress all incoming waves and eliminate spurious reflections, cf. Fig. 6. At the external boundary of $\Omega_e$ the outflow boundary condition is prescribed, cf. [1], [13], [16].

Figure 6: Square domain $\Omega$, PML layer $\Omega_e$ (shaded region) and external boundary (dashed line).

Contours of the acoustic pressure $p'(x, y, t)$ at various times are depicted in Fig. 7. The time step is $dt = 0.1$, number of points $n = 50181$, irregularly distributed and stretched in the PML. The effect of the buffer zone on incoming waves reveals clearly from Fig. 8, where the whole computational domain is depicted. Moreover, the comparison between numerical and analytical solution, cf. [1], along the line $y = 0$ is depicted in Fig. 9.
Figure 7: Acoustic pressure $p'(x, y, T)$ contours at time $T = 60, 90, 150, 210$.

Figure 8: Acoustic pressure $p'(x, y, T)$ contours at time $T = 270$. Waves moving in the PML layer are clearly dissipated and there are no spurious waves returning to the domain of interest.
5. CONCLUSION

The meshless method adjusted for the solution of linear hyperbolic systems, that are modeling e.g. the sound propagation phenomenon, was developed. Instead of the upwind approximation of derivatives, the stability of the method was achieved by the WLSQ filtering of the numerical solution after each time step. Due to the central-type local approximation of the solution in polynomial space, artificial dissipation and dispersion errors are avoided. By using this approach, higher accuracy as well as robustness of the method is reached.

The method was tested on acoustic benchmark problems with a known analytical solution, such as the 2D wall bounded acoustic pulse problem and the acoustic monopole in a free stream. Local cloud structure was created on unstructured point distributions for both domains depicted in Fig. 2 and 6. In order to suppress waves leaving the domain, the non-reflecting boundary condition in form of the PML was successfully utilized, cf. Fig. 8. Comparing the numerical and analytical results in Fig. 3, 4 and 9, very good match of both solutions can be observed.

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