A variable transformation approach for boundary element solutions of wave propagation in non-uniform potential flows

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ABSTRACT

A boundary element method in a transformed Taylor-Lorentz space-time is presented to solve sound propagation and scattering in weakly non-uniform subsonic potential flows. Boundary element solutions are conventionally provided for sound propagation and scattering in quiescent media. On the other hand, an effective approach to solve wave propagation in a non-uniform mean flow using boundary element methods has yet to be demonstrated. Although either the Taylor or Lorentz transform being applied separately has been commonly used to provide boundary integral solutions including mean flow effects on wave propagation, in this work a combination of these transformations is proposed. The Taylor-Lorentz transform allows an approximate formulation of the full potential linearized wave equation to be reduced to the standard wave equation in a deformed space-time, where a conventional boundary element method can then be devised. The boundary conditions for the formulation in the transformed space are also presented. Numerical experiments are performed to validate the present method.

Keywords: Wave propagation, Scattering, BEM, Non-Uniform flow, Taylor-Lorentz transform

1. INTRODUCTION

Due to the high computational cost and the current state of development of computer science, solving wave propagation and scattering in non-uniform flows from complex geometries is still challenging for large-scale short-wavelength problems. At the moment, boundary element methods (BEM) associated with fast multipole approaches [1] can deal with large-scale domains. However, including non-uniform flow effects in boundary element (BE) solutions is still an open problem [2].

Current boundary element modelling practices use the Lorentz transform [3–6] to solve wave propagation on a uniform mean flow. This variable transformation allows the convected Helmholtz equation to be exactly reduced to the standard Helmholtz formulation in a transformed space-time. Provided that source definitions and boundary conditions are consistent with the transformed problem, the free-field Green's function for the Helmholtz problem in quiescent media can then be used to solve sound propagation. Alternatively, a solution accounting for uniform flow effects on wave propagation based on a boundary integral formulation in the physical space, i.e. without transformation, has been proposed by Wu et al. [7] and Hu [8].

On the other hand, only approximate formulations are available to solve wave propagation in non-uniform mean flows using boundary element methods. The Taylor transformation allows a first order approximation in the mean flow Mach number of the full potential linearized wave equation to be reduced to the standard Helmholtz problem [9, 10]. Further, Tinetti and Dunn [11] have introduced a generalized local Lorentz transformation to represent the effect of non-uniform mean flows on wave propagation. The same transformation

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has been used by Clancy [12]. BE solutions based on the dual-reciprocity method (DRM) [13–15] have also been proposed. However, the absence of a robust method to define interpolating source functions restricts the application of DRM. Recently, Mancini et al. [2] have derived a boundary integral solution in the physical space, i.e. without transformation, consistent with a combination of the physical models associated with Lorentz and Taylor transforms.

Nevertheless, the Taylor-Lorentz transform [2] can be used to reduce the weakly non-uniform flow wave equation [2] to the wave operator in quiescent media and solve the boundary integral problem in the transformed space. A boundary element solution in a Taylor-Lorentz space-time has yet to be proposed to solve sound propagation in non-uniform flows and it is presented in this work. Source distribution and boundary conditions are rewritten in the transformed space and a boundary element method solving the Helmholtz problem in quiescent media is devised. In order to provide a BE kernel consistent with the standard Helmholtz problem in quiescent media [18] an iterative approach is also proposed.

The present paper is structured as follows. The numerical model is presented in Sec. 2. The Taylor-Lorentz transform is presented in Sec. 3. An integral solution in a Taylor-Lorentz space is given in Sec. 4 and the related boundary conditions are presented in Sec. 5. The boundary element formulation is derived in Sec. 6 and an iterative approach is also proposed to allow a standard BE kernel for quiescent media to be used for the prediction of non-uniform flow effects on wave propagation. Numerical results are presented in Sec. 7 to validate the present formulation.

2. NUMERICAL MODEL

The reference problem is the scattering of a sound field from a source distribution by a rigid body in a non-uniform subsonic potential mean flow (see Figure 1). Consider a homentropic, irrotational flow and acoustic perturbations of small amplitude compared with the steady mean flow. Under these assumptions, wave propagation on a non-uniform mean flow is described using a potential formulation [16] as

$$\frac{D_0}{Dt}\left(\frac{\rho_0}{c_0^2}\frac{D_0\hat{\phi}}{Dt}\right) - \boldsymbol{\nabla}\cdot\left(\rho_0\boldsymbol{\nabla}\hat{\phi}\right) = \hat{q}(\mathbf{x},t), \tag{1}$$

where $\hat{\phi}(\mathbf{x},t)$ is the acoustic velocity potential and $D_0/Dt = \partial/\partial t + \mathbf{u}_0\cdot\nabla$ denotes the material derivative over the mean flow. The mean flow density is ρ_0, c_0 is the speed of sound and $\mathbf{u}_0$ the mean flow velocity. The sound source distribution in the domain is represented by $\hat{q}(\mathbf{x},t)$.

As shown by Mancini et al. [2], the above equation can be simplified considering the dependence of the problem on the uniform mean flow Mach number M_∞, the acoustic and mean flow length scales, i.e. L_A and L_M. In the above equation consider only terms of order $[\phi]/L_A^2$, $M_\infty[\phi]/L_A^2$, $M_\infty^2[\phi]/L_A^2$ [10, 17], where $L_A \leq L_M$ and $M_\infty \ll 1$. In addition, assume a weakly non-uniform mean flow, namely a flow where the non-uniform component $\mathbf{u}_0'$ is small compared to the uniform flow part $\mathbf{u}_\infty$, given that $\mathbf{u}_0 = \mathbf{u}_0' + \mathbf{u}_\infty$. The above equation can then be rewritten as [2]

$$\frac{1}{c_\infty^2}\frac{\partial^2\hat{\phi}}{\partial t^2} + \frac{2}{c_\infty^2}\mathbf{u}_0\cdot\boldsymbol{\nabla}\frac{\partial\hat{\phi}}{\partial t} - \nabla^2\hat{\phi} + \frac{u_\infty^2}{c_\infty^2}\frac{\partial^2\hat{\phi}}{\partial x^2} = \hat{q}(\mathbf{x},t), \tag{2}$$

where u_∞ is the uniform flow velocity, which is assumed in the direction of the x-axis. Equation (2) is referred to as the weakly non-uniform potential flow wave equation. It is a first-order non-uniform flow approximation of Eq. (1) and is exact for a uniform mean flow.

Provided a source distribution $\hat{q}(\mathbf{x},t) = q(\mathbf{x})\mathrm{e}^{\mathrm{i}\omega t}$, a harmonic problem is solved, i.e. $\hat{\phi}(\mathbf{x},t) = \phi(\mathbf{x})\mathrm{e}^{\mathrm{i}\omega t}$, where q and ϕ are complex numbers. In the Fourier domain, Eq. (2) is rewritten as

$$k^2\phi - 2\mathrm{i}k\mathbf{M}_0\cdot\boldsymbol{\nabla}\phi + \nabla^2\phi - M_\infty^2\frac{\partial^2\phi}{\partial x^2} = g(\mathbf{x}), \tag{3}$$

where $k = \omega/c_\infty$, $g(\mathbf{x}) = -q(\mathbf{x})$, the Mach number $\mathbf{M}_0(\mathbf{x}) = \mathbf{u_0}(\mathbf{x})/c_\infty = \mathbf{M}_\infty + \mathbf{M}_0'(\mathbf{x})$, $\mathbf{M}_\infty = (M_\infty, 0, 0)$ and $\mathbf{M}_0'(\mathbf{x}) = (M_{0,x}', M_{0,y}', M_{0,z}')$. The problem is solved in an unbounded domain and the Sommerfeld radiation condition with mean flow needs to be satisfied for $\mathbf{x} \to \infty$. A boundary integral solution to Eq. (3) is sought.

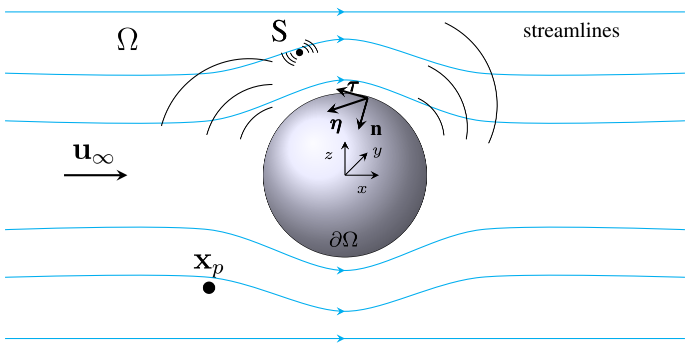


Figure 1: Geometry and main features of the reference problem: radiation and scattering of a sound field by a body in a non-uniform potential mean flow.

3. TAYLOR-LORENTZ TRANSFORM

The main features of a Taylor-Lorentz space-time transformation are presented in this section. The focus is on the transformation of the independent variables from a Taylor-Lorentz space to the physical space-time. Mancini et al. [2] have shown that Eq. (2) is associated with a physical model based on a combination of Taylor [9, 10] and Lorentz [3, 4] transforms. In other words, Eq. (2) can be reduced to the standard wave equation using a combined Taylor-Lorentz transform, that is

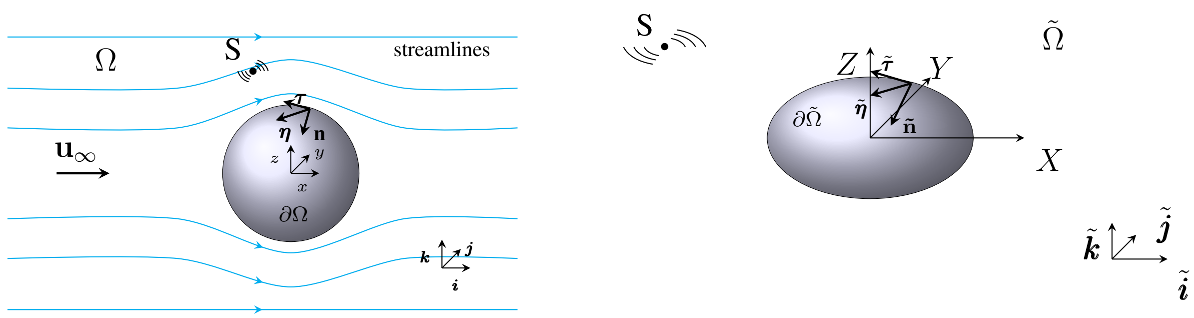
$$X = \frac{x}{\beta_\infty}, \quad Y = y, \quad Z = z, \quad T = \beta_\infty\left(t + \frac{\Phi_0'(\mathbf{x})}{c_\infty^2}\right) + \frac{M_\infty x}{c_\infty \beta_\infty}, \tag{4}$$

where $(\mathbf{X}, T)$ and $(\mathbf{x}, t)$ denote respectively the transformed and the physical space-time coordinates, $\Phi_0'(\mathbf{x})$ is the velocity potential associated with the mean flow non-uniform component and $\beta_\infty = \sqrt{1 - M_\infty^2}$. Using the chain rule based on Eq. (4), the derivative operators can then be rewritten as

$$\nabla_X = \left(\beta_\infty \frac{\partial}{\partial x} - \frac{1}{c_\infty}\left[\frac{M_\infty}{\beta_\infty} + \beta_\infty M_{0,x}'\right]\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial y} - \frac{M_{0,y}'}{c_\infty}\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial z} - \frac{M_{0,z}'}{c_\infty}\frac{\partial}{\partial t}\right), \quad \frac{\partial}{\partial T} = \frac{1}{\beta_\infty}\frac{\partial}{\partial t}. \tag{5}$$

For a harmonic problem, the above equation can be written considering $\partial/\partial t = \mathrm{i}\omega$.

Denote the orthonormal bases in the physical and in the transformed space respectively as $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ and $(\tilde{\mathbf{i}}, \tilde{\mathbf{j}}, \tilde{\mathbf{k}})$ (see Figure 2). Using Eq. (4) and defining the normal vector $\mathbf{n} = n_x\mathbf{i} + n_y\mathbf{j} + n_z\mathbf{k}$ to the boundary surface $\partial\Omega$ (see Figure 2), the corresponding vector in the transformed space $\tilde{\mathbf{n}} = n_X\tilde{\mathbf{i}} + n_Y\tilde{\mathbf{j}} + n_Z\tilde{\mathbf{k}}$ is given as $\tilde{\mathbf{n}} = (\beta_\infty n_x\mathbf{i} + n_y\mathbf{j} + n_z\mathbf{k})/\sqrt{1 - M_\infty^2 n_x^2}$. Note that $\tilde{\mathbf{n}}$ is also a unit vector in the transformed space if measured with the Taylor-Lorentz metrics.



(a) Physical domain

(b) Taylor-Lorentz domain

Figure 2: Geometrical features for the scattering of a sound field in a non-uniform potential flow. Description of the same problem either in a physical or in a transformed Taylor-Lorentz space-time.

4. BOUNDARY INTEGRAL SOLUTION

A boundary integral solution to Eq. (3) is sought in a Taylor-Lorentz space. Using Eqs. (4) and (5), Eq. (2) is reduced to the Helmholtz equation [2] in quiescent media

$$\frac{1}{c_\infty^2}\frac{\partial^2\bar{\phi}}{\partial T^2} - \nabla_X^2\bar{\phi} = \bar{q}(\mathbf{X},T), \tag{6}$$

where $\bar{\phi} = \bar{\phi}(\mathbf{X},T)$. In the physical space, a harmonic solution $\hat{\phi}(\mathbf{x},t) = \phi(\mathbf{x})\mathrm{e}^{\mathrm{i}\omega t}$ due a harmonic source $\hat{q}(\mathbf{x},t) = q(\mathbf{x})\mathrm{e}^{\mathrm{i}\omega t}$ is sought. In the transformed space we also look for a harmonic solution $\bar{\phi}(\mathbf{X},T) = \tilde{\phi}(\mathbf{X})\mathrm{e}^{\mathrm{i}\tilde{\omega}T}$ given a harmonic source $\bar{q}(\mathbf{X},T) = \tilde{q}(\mathbf{X})\mathrm{e}^{\mathrm{i}\tilde{\omega}T}$. Using Eq. (4) and giving $\tilde{g}(\mathbf{X}) = -\tilde{q}(\mathbf{X})$ one has

$$\tilde{\phi}(\mathbf{X}) = \phi(\mathbf{x})\mathrm{e}^{-\mathrm{i}\omega\left(\frac{M_\infty x}{c_\infty\beta_\infty^2}+\frac{\Phi_0'(\mathbf{x})}{c_\infty^2}\right)}, \tag{7}$$

$$\tilde{g}(\mathbf{X}) = g(\mathbf{x})\mathrm{e}^{-\mathrm{i}\omega\left(\frac{M_\infty x}{c_\infty\beta_\infty^2}+\frac{\Phi_0'(\mathbf{x})}{c_\infty^2}\right)}. \tag{8}$$

Rewrite Eq. (6) for a harmonic problem using Eq. (8) to give

$$\nabla_X^2\tilde{\phi} + \tilde{k}^2\tilde{\phi} = \tilde{g}, \tag{9}$$

where $\tilde{k} = k/\beta_\infty$ and the superscript "$\sim$" denotes the dependent variables in the transformed space. The above equation is nothing but the non-homogeneous Helmholtz equation in the transformed Taylor-Lorentz space-time.

For the non-homogeneous Helmholtz equation, Eq. (9), a boundary integral solution is well-known [19]

$$C(\mathbf{X}_p)\tilde{\phi}(\mathbf{X}_p) = \int_{\partial\tilde{\Omega}}\left[\tilde{G}(\mathbf{X}_p,\mathbf{X})\frac{\partial\tilde{\phi}(\mathbf{X})}{\partial\tilde{n}} - \tilde{\phi}(\mathbf{X})\frac{\partial\tilde{G}(\mathbf{X}_p,\mathbf{X})}{\partial\tilde{n}}\right]dS + \int_{\tilde{\Omega}}\tilde{G}(\mathbf{X}_p,\mathbf{X})\tilde{g}(\mathbf{X})\,dV, \tag{10}$$

with

$$C(\mathbf{X}_p) = \begin{cases} 1 & \mathbf{X}_p \in \tilde{\Omega} \\ 1 - \int_{\partial\tilde{\Omega}}\frac{\partial\tilde{G}_0}{\partial\tilde{n}}dS(\mathbf{X}) & \mathbf{X}_p \in \partial\tilde{\Omega} \\ 0 & \mathbf{X}_p \notin \partial\tilde{\Omega}\cup\tilde{\Omega} \end{cases}, \tag{11}$$

where $\tilde{\Omega}$ and $\partial\tilde{\Omega}$ denote the transformed space and its boundary. Note that $\tilde{G}(\mathbf{X}_p,\mathbf{X})$ in Eq. (10) is the free-field Green's function associated with the Helmholtz problem for quiescent media, i.e.

$$\tilde{G}(\mathbf{X}_p,\mathbf{X}) = \frac{\mathrm{e}^{\mathrm{i}\tilde{k}\tilde{R}}}{4\pi\tilde{R}}, \tag{12}$$

where $\tilde{R} = \sqrt{(X_p-X)^2+(Y_p-Y)^2+(Z_p-Z)^2}$, with $\mathbf{X}_p$ and $\mathbf{X}$ respectively the position of the observer and the source point. On the other hand, $\tilde{G}_0$ is the Green's function associated with the Laplace operator, $\nabla_{\tilde{X}}^2\tilde{G}_0 = \delta(\mathbf{X}-\mathbf{X}_p)$ written in the transformed space [7], namely $\tilde{G}_0(\mathbf{X}_p,\mathbf{X}) = 1/(4\pi\tilde{R})$. The Green's function for the 2D problem is also well-known [19], but it is not reported for conciseness. After solving Eq. (10) along $\partial\tilde{\Omega}$ and computing $\tilde{\phi}$ in the transformed domain $\tilde{\Omega}$, the solution ϕ in the physical domain can be obtained using Eq. (7).

5. BOUNDARY CONDITIONS

Dirichlet and Neumann boundary conditions associated with Eq. (3) are presented in a transformed Taylor-Lorentz space. If a Dirichlet problem is solved, the condition on the boundary surface is generally expressed by $\phi(\mathbf{x}) = f(\mathbf{x})$ for $\mathbf{x} \in \partial\Omega$, where $f(\mathbf{x})$ is a known function. Using Eq. (7) gives

$$\tilde{\phi}(\mathbf{X}) = f(\mathbf{x})\mathrm{e}^{-\mathrm{i}k\left(\frac{M_\infty x}{\beta_\infty^2}+\frac{\Phi_0'(\mathbf{x})}{c_\infty}\right)} = \tilde{f}(\mathbf{X}). \tag{13}$$

On the other hand, the Neumann problem is given by prescribing $\partial\phi(\mathbf{x})/\partial n = h_n(\mathbf{x})$ for $\mathbf{x} \in \partial\Omega$, where $\mathbf{n}$ is the outgoing normal vector to the boundary surface $\partial\Omega$ (see Figures 1) and h_n is a well-known function. The acoustic particle velocity $\partial\phi(\mathbf{x})/\partial n$ can be written in the Taylor-Lorentz space deriving Eq. (7) along the normal direction $\mathbf{n}$. Assume $M_0' \ll M_\infty \ll 1$, such that $M_\infty n_x + M_{0,n}'\beta_\infty^2 = M_\infty n_x + M_{0,n}'(1 - M_\infty^2) \simeq M_\infty n_x + M_{0,n}' = M_{0,n}$, to give

$$\frac{\partial\phi(\mathbf{x})}{\partial n} = \left[\frac{\partial\tilde{\phi}(\mathbf{X})}{\partial n} + \mathrm{i}\frac{k}{\beta_\infty^2}M_{0,n}\tilde{\phi}\right]\mathrm{e}^{\mathrm{i}k\left(\frac{M_\infty x}{\beta_\infty^2} + \frac{\Phi_0'(\mathbf{x})}{c_\infty}\right)}. \tag{14}$$

For an impermeable surface, such as $\partial\Omega$ (see Figure 1), the normal component of the mean flow to the boundary surface $\partial\Omega$ is zero, i.e. $M_{0,n} = 0$. Hence, Eq. (14) can be reduced to

$$\frac{\partial\tilde{\phi}(\mathbf{X})}{\partial n} = h_n(\mathbf{x})\mathrm{e}^{-\mathrm{i}k\left(\frac{M_\infty x}{\beta_\infty^2} + \frac{\Phi_0'(\mathbf{x})}{c_\infty}\right)}. \tag{15}$$

Note that the normal vector $\mathbf{n}$ to the boundary surface $\partial\Omega$ is not normal to the boundary surface in the transformed domain $\partial\tilde{\Omega}$ (see Figure 2). So, define the tangent vector to the transformed boundary surface $\tilde{\boldsymbol{\tau}}$ such that, given the normal vector $\tilde{\boldsymbol{n}}$ to the same surface, an orthonormal triplet can be assigned as $\tilde{\boldsymbol{\eta}} = \tilde{\boldsymbol{\tau}} \times \tilde{\boldsymbol{n}}$ (see Figure 2). The definition of $\tilde{\boldsymbol{n}}$, $\tilde{\boldsymbol{\tau}}$ and $\tilde{\boldsymbol{\eta}}$ allows $\partial\tilde{\phi}(\mathbf{X})/\partial n$ to be rewritten as

$$\frac{\partial\tilde{\phi}}{\partial n} = \frac{\partial\tilde{\phi}}{\partial\tilde{n}}\frac{\partial\tilde{n}}{\partial n} + \frac{\partial\tilde{\phi}}{\partial\tilde{\tau}}\frac{\partial\tilde{\tau}}{\partial n} + \frac{\partial\tilde{\phi}}{\partial\tilde{\eta}}\frac{\partial\tilde{\eta}}{\partial n}. \tag{16}$$

Then, define $\mu = 1/\left(\partial\tilde{n}/\partial n\right)$ and apply Eq. (15) to Eq. (16) giving

$$\frac{\partial\tilde{\phi}(\mathbf{X})}{\partial\tilde{n}} = \mu\left[h_n(\mathbf{x})\mathrm{e}^{-\mathrm{i}k\left(\frac{M_\infty x}{\beta_\infty^2} + \frac{\Phi_0'(\mathbf{x})}{c_\infty}\right)} - \frac{\partial\tilde{\phi}(\mathbf{X})}{\partial\tilde{\tau}}\frac{\partial\tilde{\tau}}{\partial n} - \frac{\partial\tilde{\phi}(\mathbf{X})}{\partial\tilde{\eta}}\frac{\partial\tilde{\eta}}{\partial n}\right]. \tag{17}$$

Consistent with Eq. (7), the above equation can be rewritten by defining

$$\tilde{h}_n(\mathbf{X}) = h_n(\mathbf{x})\mathrm{e}^{-\mathrm{i}k\left(\frac{M_\infty x}{\beta_\infty^2} + \frac{\Phi_0'(\mathbf{x})}{c_\infty}\right)}, \tag{18}$$

so that

$$\frac{\partial\tilde{\phi}(\mathbf{X})}{\partial\tilde{n}} = \mu\left[\tilde{h}_n(\mathbf{X}) - \frac{\partial\tilde{\phi}(\mathbf{X})}{\partial\tilde{\tau}}\frac{\partial\tilde{\tau}}{\partial n} - \frac{\partial\tilde{\phi}(\mathbf{X})}{\partial\tilde{\eta}}\frac{\partial\tilde{\eta}}{\partial n}\right]. \tag{19}$$

Note that for rigid body scattering $\partial\phi/\partial n = h_n = 0$, namely $\tilde{h}_n = 0$. Equations (13) and (17) allow Eq. (10) to be also written for mixed boundary conditions.

6. BOUNDARY ELEMENT FORMULATION

6.1 Variational formulation

A variational formulation of Eq. (10) written along $\partial\tilde{\Omega}$ is presented based on a Burton–Miller [20] approach. The Burton-Miller formulation is adopted in the transformed Taylor-Lorentz space to overcome the irregular frequency issue affecting boundary element solutions [18]. Given Eq. (10), define an incident field $\tilde{\phi}_i$ as

$$\tilde{\phi}_i(\mathbf{X}_p) = \int_{\tilde{\Omega}} \tilde{G}(\mathbf{X}_p, \mathbf{X})\tilde{g}(\mathbf{X})dV. \tag{20}$$

Assign $C(\mathbf{X}_p) = 1/2$ for $\mathbf{X}_p \in \partial\Omega$ as conventionally done for regular surfaces [22], and consider the Burton-Miller formulation [20]. Then, introduce a test function $\tilde{\phi}^*$ to give

$$\begin{aligned}\frac{1}{2}\int_{\partial\tilde{\Omega}} \tilde{\phi}_p^*\left[\tilde{\phi}_p + \alpha\frac{\partial\tilde{\phi}_p}{\partial\tilde{n}_p}\right]dS_p = &\int_{\partial\tilde{\Omega}}\int_{\partial\tilde{\Omega}} \tilde{\phi}_p^*\left[\tilde{G}\frac{\partial\tilde{\phi}}{\partial\tilde{n}} - \tilde{\phi}\frac{\partial\tilde{G}}{\partial\tilde{n}} + \alpha\left(\frac{\partial\tilde{G}}{\partial\tilde{n}_p}\frac{\partial\tilde{\phi}}{\partial\tilde{n}} - \tilde{\phi}\frac{\partial^2\tilde{G}}{\partial\tilde{n}\partial\tilde{n}_p}\right)\right]dSdS_p \\ &+ \int_{\partial\tilde{\Omega}} \tilde{\phi}_p^*\left[\tilde{\phi}_{i,p} + \alpha\frac{\partial\tilde{\phi}_{i,p}}{\partial\tilde{n}_p}\right]dS_p,\end{aligned} \tag{21}$$

where $\tilde{\mathbf{n}}_p = \tilde{\mathbf{n}}(\mathbf{X}_p)$ is the normal vector to the boundary surface $\partial\tilde{\Omega}$ (see Figure 2), $\tilde{\phi} = \tilde{\phi}(\mathbf{X})$, $\tilde{\phi}_{i,p} = \tilde{\phi}_i(\mathbf{X}_p)$ and $\tilde{\phi}_p = \tilde{\phi}(\mathbf{X}_p)$ with $\tilde{\phi}^*$ the complex conjugate of $\tilde{\phi}$. In the above equation $dS_p = dS(X_p, Y_p, Z_p)$ and $dS = dS(X, Y, Z)$. The coefficient α is given following the work of Burton and Miller [20] as $\alpha = i/\tilde{k}$. In the above equation the hyper-singular integral including $\partial^2\tilde{G}/\partial\tilde{n}\partial\tilde{n}_p$ can be regularized using the solution proposed by Hamdi [21] for a variational formulation,

$$\int_{\partial\tilde{\Omega}}\int_{\partial\tilde{\Omega}} \tilde{\phi}_p^* \frac{\partial^2 \tilde{G}}{\partial\tilde{n}\partial\tilde{n}_p}\tilde{\phi} d\tilde{S} d\tilde{S}_p = \int_{\partial\tilde{\Omega}}\int_{\partial\tilde{\Omega}} \tilde{G}\left[\tilde{k}^2 \tilde{\mathbf{n}}\cdot\tilde{\mathbf{n}}_p \tilde{\phi}\tilde{\phi}_p^* - \left(\tilde{\mathbf{n}}_p \times \boldsymbol{\nabla}_X \tilde{\phi}_p^*\right)\cdot\left(\tilde{\mathbf{n}}\times\boldsymbol{\nabla}_X\tilde{\phi}\right)\right] d\tilde{S} d\tilde{S}_p. \quad (22)$$

Substituting Eq. (22) in Eq. (21) gives an equation with only non-singular and weakly singular integrals.

A discrete system of equations in the case of either a Dirichlet or Neumann problem is sought. The discrete system of equations for mixed boundary condition can then be obtained from a combination of the former and the latter. After solving the problem on $\partial\tilde{\Omega}$, Eq. (10) can be used to compute the solution in $\tilde{\Omega}$. The solution in the physical space Ω is given by using Eq. (7).

6.2 Dirichlet problem

The Dirichlet problem associated with Eq. (3) can be solved based on Eqs. (13) and (21). The dependent variables in Eq. (21) are approximated by

$$\tilde{\phi}(\mathbf{X}) = \sum_{r=1}^{NDoF} N_r(\mathbf{X})\tilde{\phi}_r, \qquad \frac{\partial\tilde{\phi}(\mathbf{X})}{\partial\tilde{n}} = \sum_{r=1}^{NDoF} N_r(\mathbf{X})\frac{\partial\tilde{\phi}_r}{\partial\tilde{n}}, \quad (23)$$

where $N_r(\mathbf{X})$ is the r-th polynomial shape function and $NDoF$ is the total number of degrees of freedom. Use for $\tilde{\phi}^*$ the same discretisation used for ϕ in the above equation to give

$$(\boldsymbol{H} + \alpha\hat{\boldsymbol{H}})\frac{\partial\tilde{\boldsymbol{\phi}}}{\partial\tilde{n}} = -(\boldsymbol{D} + \alpha\hat{\boldsymbol{D}})\tilde{\boldsymbol{f}} + \boldsymbol{F} + \alpha\hat{\boldsymbol{F}} \quad (24)$$

where $\tilde{f}(\mathbf{X}) = \sum_{r=1}^{NDoF} N_r(\mathbf{X})\tilde{f}_r$, $\tilde{\boldsymbol{f}} = \{f_1, f_2, ..., f_{NDoF}\}$ the vector of the boundary conditions written at nodal points and

$$D^{ij} = \frac{1}{2}\int_{\partial\tilde{\Omega}} \delta_{ij} N_i^*(\mathbf{X}_p) N_j(\mathbf{X}_p) dS_p + \int_{\partial\tilde{\Omega}}\int_{\partial\tilde{\Omega}} N_i^*(\mathbf{X}_p)\frac{\partial\tilde{G}(\mathbf{X}_p,\mathbf{X})}{\partial\tilde{n}} N_j(\mathbf{X}) dS dS_p, \quad (25)$$

$$\hat{D}^{ij} = \int_{\partial\tilde{\Omega}}\int_{\partial\tilde{\Omega}} N_i^*(\mathbf{X}_p)\frac{\partial^2\tilde{G}(\mathbf{X}_p,\mathbf{X})}{\partial\tilde{n}\partial\tilde{n}_p} N_j(\mathbf{X}) dS dS_p, \quad (26)$$

$$H^{ij} = -\int_{\partial\tilde{\Omega}}\int_{\partial\tilde{\Omega}} N_i^*(\mathbf{X}_p)\tilde{G}(\mathbf{X}_p,\mathbf{X}) N_j(\mathbf{X}) dS dS_p, \quad (27)$$

$$\hat{H}^{ij} = \frac{1}{2}\int_{\partial\tilde{\Omega}} \delta_{ij} N_i^*(\mathbf{X}_p) N_j(\mathbf{X}_p) dS_p - \int_{\partial\tilde{\Omega}}\int_{\partial\tilde{\Omega}} N_i^*(\mathbf{X}_p)\frac{\partial\tilde{G}(\mathbf{X}_p,\mathbf{X})}{\partial\tilde{n}_p} N_j(\mathbf{X}) dS dS_p, \quad (28)$$

$$F^i = \int_{\partial\tilde{\Omega}} N_i^*(\mathbf{X}_p)\tilde{\phi}_{i,p} dS_p, \qquad \hat{F}^i = \int_{\partial\tilde{\Omega}} N_i^*(\mathbf{X}_p)\frac{\partial\tilde{\phi}_{i,p}}{\partial\tilde{n}_p} dS_p. \quad (29)$$

Note that the hypersingular integral in Eq. (21) can be regularized using Eq. (22).

6.3 Neumann problem

The Neumann problem associated with Eq. (3) can be solved by using Eq. (19) and Eq. (21). First, approximate the tangential derivatives in Eq. (19) as

$$\frac{\partial\tilde{\phi}(\mathbf{X})}{\partial\tilde{\tau}} = \sum_{r=1}^{NDoF} \frac{\partial N_r(\mathbf{X})}{\partial\tilde{\tau}}\tilde{\phi}_r, \qquad \frac{\partial\tilde{\phi}(\mathbf{X})}{\partial\tilde{\eta}} = \sum_{r=1}^{NDoF} \frac{\partial N_r(\mathbf{X})}{\partial\tilde{\eta}}\tilde{\phi}_r. \tag{30}$$

Then, use Eqs. (23) and (30) in Eq. (21) to give

$$(\bar{\boldsymbol{D}} + \alpha\hat{\bar{\boldsymbol{D}}})\tilde{\boldsymbol{\phi}} = -(\bar{\boldsymbol{H}} + \alpha\hat{\bar{\boldsymbol{H}}})\tilde{\boldsymbol{h}}_n + \boldsymbol{F} + \alpha\hat{\boldsymbol{F}}, \tag{31}$$

where

$$\begin{aligned}\bar{D}^{ij} &= \frac{1}{2}\int_{\partial\tilde{\Omega}} \delta_{ij} N_i^*(\mathbf{X}_p) N_j(\mathbf{X}_p) dS_p + \int_{\partial\tilde{\Omega}}\int_{\partial\tilde{\Omega}} N_i^*(\mathbf{X}_p)\frac{\partial\tilde{G}(\mathbf{X}_p,\mathbf{X})}{\partial\tilde{n}} N_j(\mathbf{X}) dS dS_p \\ &+ \int_{\partial\tilde{\Omega}}\int_{\partial\tilde{\Omega}} \mu_j N_i^*(\mathbf{X}_p)\tilde{G}(\mathbf{X}_p,\mathbf{X})\left[\frac{\partial N_j(\mathbf{X})}{\partial\tilde{\tau}}\frac{\partial\tilde{\tau}}{\partial n} + \frac{\partial N_j(\mathbf{X})}{\partial\tilde{\eta}}\frac{\partial\tilde{\eta}}{\partial n}\right] dS dS_p,\end{aligned} \tag{32}$$

$$\begin{aligned}\hat{\bar{D}}^{ij} &= -\frac{1}{2}\int_{\partial\tilde{\Omega}} \mu_j\delta_{ij} N_i^*(\mathbf{X}_p)\left[\frac{\partial N_j(\mathbf{X}_p)}{\partial\tilde{\tau}}\frac{\partial\tilde{\tau}}{\partial n} + \frac{\partial N_j(\mathbf{X}_p)}{\partial\tilde{\eta}}\frac{\partial\tilde{\eta}}{\partial n}\right] dS_p \\ &+ \int_{\partial\tilde{\Omega}}\int_{\partial\tilde{\Omega}} N_i^*(\mathbf{X}_p)\frac{\partial^2\tilde{G}(\mathbf{X}_p,\mathbf{X})}{\partial\tilde{n}\partial\tilde{n}_p} N_j(\mathbf{X}) dS dS_p \\ &+ \int_{\partial\tilde{\Omega}}\int_{\partial\tilde{\Omega}} \mu_j N_i^*(\mathbf{X}_p)\frac{\partial\tilde{G}(\mathbf{X}_p,\mathbf{X})}{\partial\tilde{n}_p}\left[\frac{\partial N_j(\mathbf{X})}{\partial\tilde{\tau}}\frac{\partial\tilde{\tau}}{\partial n} + \frac{\partial N_j(\mathbf{X})}{\partial\tilde{\eta}}\frac{\partial\tilde{\eta}}{\partial n}\right] dS dS_p,\end{aligned} \tag{33}$$

the vector $\tilde{\boldsymbol{h}}_n$ is given approximating Eq. (18) as $\tilde{h}_n(\mathbf{X}) = \sum_{r=1}^{NDoF} N_r(\mathbf{X})\tilde{h}_{n,r}$, $\boldsymbol{F}$ and $\hat{\boldsymbol{F}}$ are given in Eq. (29) and $\mu_j = \mu_j(\mathbf{X})$. The hypersingular integral in Eq. (33) can also be regularized using Eq. (22). In addition,

$$\bar{H}^{ij} = -\int_{\partial\tilde{\Omega}}\int_{\partial\tilde{\Omega}} \mu_j N_i^*(\mathbf{X}_p)\tilde{G}(\mathbf{X}_p,\mathbf{X}) N_j(\mathbf{X}) dS dS_p, \tag{34}$$

$$\hat{\bar{H}}^{ij} = \frac{1}{2}\int_{\partial\tilde{\Omega}} \mu_j\delta_{ij} N_i^*(\mathbf{X}_p) N_j(\mathbf{X}_p) dS_p - \int_{\partial\tilde{\Omega}}\int_{\partial\tilde{\Omega}} \mu_j N_i^*(\mathbf{X}_p)\frac{\partial\tilde{G}(\mathbf{X}_p,\mathbf{X})}{\partial\tilde{n}_p} N_j(\mathbf{X}) dS dS_p. \tag{35}$$

6.4 Iterative solution for standard boundary element kernels

Although the l.h.s. of Eq. (24) is compatible with a standard BE kernel for the Helmholtz problem in quiescent media [18], the l.h.s. of Eq. (31) includes the derivatives of $\tilde{\phi}$ along the vectors $\tilde{\boldsymbol{\tau}}$ and $\tilde{\boldsymbol{\eta}}$ (see Figure 1). In order to solve the system of equations Eq. (31) using a standard BE kernel, an iterative approach is presented. This approach is convenient because it allows efficient BE algorithms, such as the fast multipole BEM [1], to solve wave propagation on a non-uniform mean flow, provided that geometry, boundary conditions and source distribution are written in a Taylor-Lorentz space.

We represent the integrals including the tangent derivatives of $\tilde{\phi}$ as equivalent sources along the boundary surface. So, we bring the tangent derivatives of $\tilde{\phi}$ to the r.h.s. of Eq. (31) and set up an iterative solution. Since the source terms along the boundary depends on $\tilde{\phi}$, the distribution of sources along $\partial\tilde{\Omega}$ is updated based on the solution $\tilde{\phi}$ at a previous iterative step (see Figure 3). Therefore, Eq. (31) is rewritten as

$$(\boldsymbol{D} + \alpha\hat{\boldsymbol{D}})\tilde{\boldsymbol{\phi}}_{m+1} = (\bar{\boldsymbol{H}} + \alpha\hat{\bar{\boldsymbol{H}}})\tilde{\boldsymbol{h}}_n + \boldsymbol{V}_m + \alpha\hat{\boldsymbol{V}}_m, \tag{36}$$

where $m = 0, 1...N$ denotes the m-th iterative step and $\boldsymbol{D}$, $\hat{\boldsymbol{D}}$ are given in Eqs. (25) and (26),

$$V_m^i = \int_{\partial\tilde{\Omega}} N_i^*(\mathbf{X}_p)\tilde{\phi}_{i,p} dS_p - \int_{\partial\tilde{\Omega}}\int_{\partial\tilde{\Omega}} \mu_j N_i^*(\mathbf{X}_p) G(\mathbf{X}_p,\mathbf{X})\left[\frac{\partial N_j(\mathbf{X})}{\partial\tilde{\tau}}\frac{\partial\tilde{\tau}}{\partial n} + \frac{\partial N_j(\mathbf{X})}{\partial\tilde{\eta}}\frac{\partial\tilde{\eta}}{\partial n}\right]\tilde{\phi}_{j,m} dS dS_p \tag{37}$$

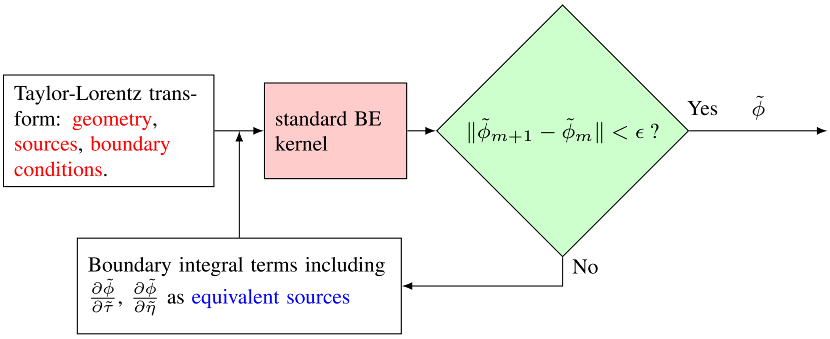


Figure 3: Schematic representation of the iterative scheme to use standard boundary element kernels to solve sound propagation in a non-uniform flow based on a Taylor-Lorentz transformation.

$$\begin{aligned}\hat{V}^i_m = &\int_{\partial\tilde{\Omega}} N_i^*(\mathbf{X}_p)\frac{\partial\tilde{\phi}_{i,p}}{\partial\tilde{n}_p}dS_p + \frac{1}{2}\int_{\partial\tilde{\Omega}} \mu_j\delta_{ij}N_i^*(\mathbf{X}_p)\left[\frac{\partial N_j(\mathbf{X})}{\partial\tilde{\tau}}\frac{\partial\tilde{\tau}}{\partial n} + \frac{\partial N_j(\mathbf{X})}{\partial\tilde{\eta}}\frac{\partial\tilde{\eta}}{\partial n}\right]\tilde{\phi}_{j,m}dS_p \\ &- \int_{\partial\tilde{\Omega}}\int_{\partial\tilde{\Omega}} \mu_j N_i^*(\mathbf{X}_p)\frac{\partial\tilde{G}(\mathbf{X}_p,\mathbf{X})}{\partial\tilde{n}_p}\left[\frac{\partial N_j(\mathbf{X})}{\partial\tilde{\tau}}\frac{\partial\tilde{\tau}}{\partial n} + \frac{\partial N_j(\mathbf{X})}{\partial\tilde{\eta}}\frac{\partial\tilde{\eta}}{\partial n}\right]\tilde{\phi}_{j,m}dSdS_p.\end{aligned} \tag{38}$$

In the above equations $\mu_j = \mu_j(\mathbf{X})$, $\bar{H}^{ij}$ and $\hat{\bar{H}}^{ij}$ are given in Eqs. (34) and (35), and $\phi_{j,m}$ denotes the j-th nodal value of $\tilde{\phi}$ at the m-th iteration. A convergence criterion for the iterative scheme based on a tolerance ϵ measuring $\|\tilde{\boldsymbol{\phi}}_{m+1} - \tilde{\boldsymbol{\phi}}_m\|$ can then be established. As a first step, we solve the system of equations Eq. (31) without including the tangent derivatives, namely for $m = 0$ we impose

$$V^i_0 = \int_{\partial\tilde{\Omega}} N_i^*(\mathbf{X}_p)\tilde{\phi}_i dS_p, \qquad \hat{V}^i_0 = \int_{\partial\tilde{\Omega}} N_i^*(\mathbf{X}_p)\frac{\partial\tilde{\phi}_i}{\partial\tilde{n}_p}dS_p. \tag{39}$$

7. NUMERICAL RESULTS

Scattering of a sound field by a 2D rigid cylinder of radius a in a non-uniform subsonic potential mean flow is used as a test case. The solution obtained solving the boundary element problem in the Taylor-Lorentz space is compared to the results obtained solving the same physical model as Eq. (3) but using an integral solution in the physical space [2]. A Lagrangian BE solution based on cubic interpolation is given. A cubic FE solution of the reference physical model, i.e. Eq. (1), is also provided. In either case, we consider a mesh resolution with 8 degrees of freedom per wavelength. A mean flow is described using the analytical solution of an incompressible potential flow around a cylinder without circulation. The speed of sound is $c_\infty = 340$ m/s, the mean flow density is $\rho_\infty = 1.22$ kg/m^3. The sound pressure level is computed as SPL$= 20\log_{10}(p_{rms}/p_{ref})$ where $p_{ref} = 2\cdot 10^{-5}$ Pa, $p_{rms} = \|p\|/\sqrt{2}$ and $p = -\rho_\infty(i\omega\phi + \mathbf{u}_0\cdot\nabla\phi)$.

7.1 Assessment of the BE solution

First, a BE solution imposing $\alpha = 0$ in Eq. (21) is provided solving the problem in Figure 4 below the first resonant frequency of the cylinder, that is $ka \sim 2.22$. A distribution of point sources is located on a line with origin at $\mathbf{x}_s = (-18a, 4a)$, parallel to the y-axis with length $D = 3a$ (see Figure 4). The source is defined using 15 equally distributed monopole point sources of unit magnitude and adjusting the amplitude with a weight function,

$$w(y) = \cos(\pi n y/D) \tag{40}$$

where $0 \le y \le D$. A unit source amplitude is retrieved using a scaling factor $1/\int_0^D \|w(y)\|dy$. Consider $n = 3$ in the above equation. In Figure 5 the real and imaginary part of the ϕ along the surface of the cylinder, i.e. $r = a$, are plotted for a non-dimensional frequency $ka = 2$ and $M_\infty = 0.1$ and 0.3, comparing the results obtained using Eq. (31) with $\alpha = 0$ and the results obtained using the formulation of Mancini et al. [2]. The angle θ is zero along the $x-$axis and measured counterclockwise (see Figure 4).

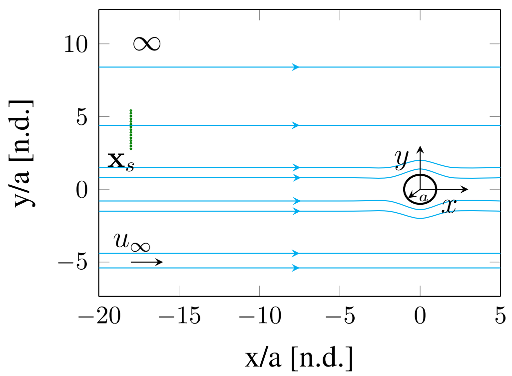
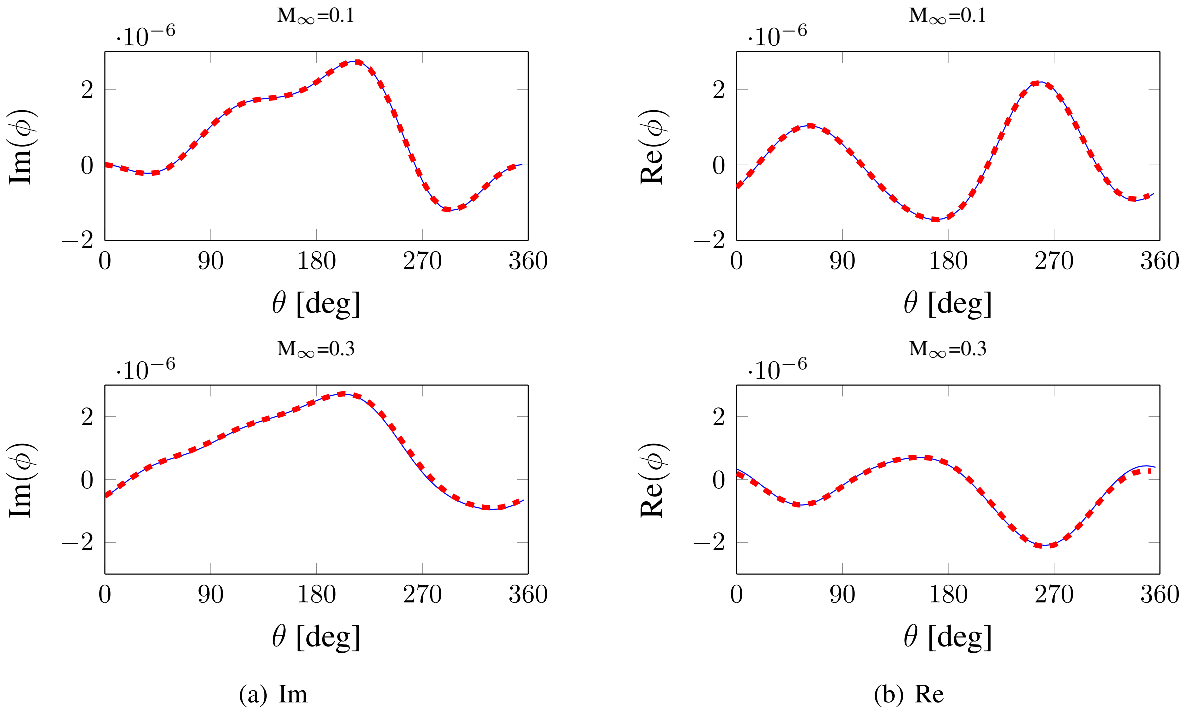


Figure 4: Geometry and main features for the scattering of the sound field generated from a source distribution by a rigid cylinder ($\partial\phi/\partial n = 0$) in a non-uniform potential subsonic flow.



(a) Im (b) Re

Figure 5: Real and imaginary part of ϕ along an arc of field points with radius $r = a$ for a non-dimensional frequency $ka = 2$, $n = 3$ in Eq. (40) and the problem described in Figure 4. BE physical space [2] (solid), BE transformed space (see Eq. (31)) (dashed).

Secondly, we consider a monopole point source at $\mathbf{x}_s = (-4a, -4a)$ and solve the problem in Figure 6. The sound pressure level (SPL) is depicted in Figure 7 along an arc of field points with radius $r = 8a$ for non-dimensional frequencies $ka = 4$ and 16, $M_\infty = 0.1$ and 0.3. In Figure 7 we compare the BE solution based on the Burton–Miller formulation either in the physical [23] ("*BE phys.*") or in the transformed domain (see Eq. (31)) ("*BE transf.*") with a FE solution of the reference physical model (see Eq. (1)) ("*FE ref.*") written for a harmonic problem. The phase of the pressure field along the same arc of field points is shown in Figure 8 at a non-dimensional frequency $ka = 16$ and $M_\infty = 0.1, 0.3$.

The numerical results given by the BE solution in the Taylor-Lorentz space are in good agreement with the results obtained using the corresponding integral solution in the physical space (see Figure 5). A good agreement is also shown between the Burton–Miller formulation written either in the physical [23] or transformed domain (see Figure 7-8). Although both the formulations are consistent with Eq. (3), a small difference is shown because of the approximation of boundary conditions and sources. However, this difference is small compared with the accuracy of Eq. (2) against the reference physical model Eq. (1) [2].

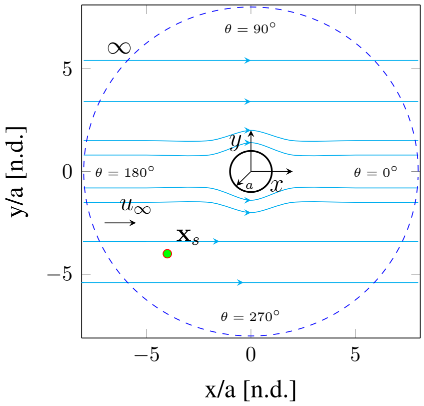
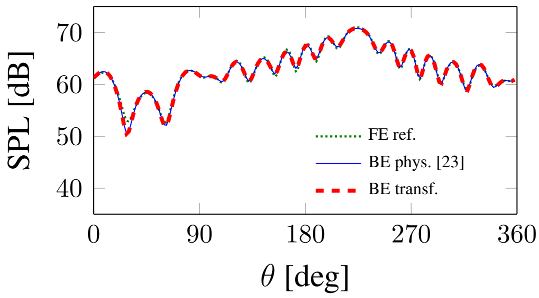
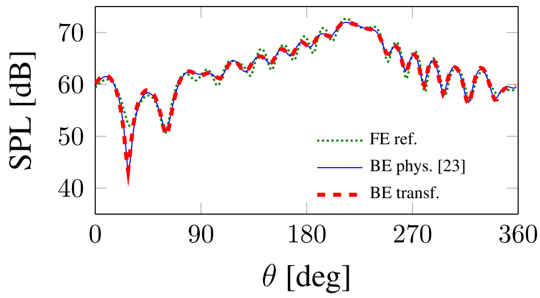


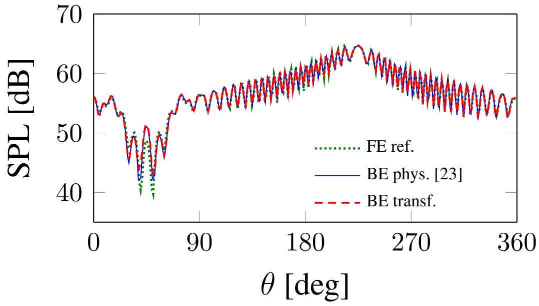
Figure 6: Geometry and main features of the reference problem: scattering of the sound field generated from a monopole point source by a rigid cylinder ($\partial\phi/\partial n = 0$) in a non-uniform potential subsonic flow.



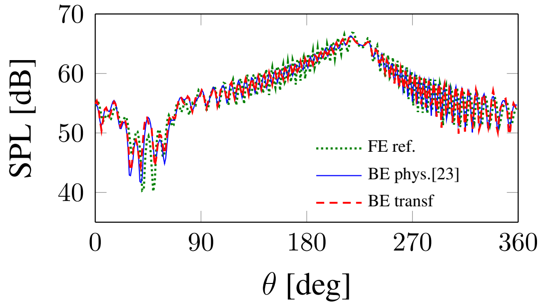
(a) $M_\infty = 0.1$, $ka = 4$



(b) $M_\infty = 0.3$, $ka = 4$

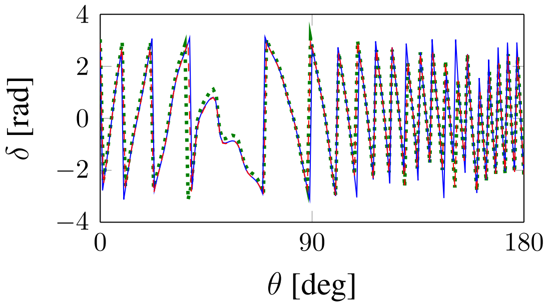


(c) $M_\infty = 0.1$, $ka = 16$

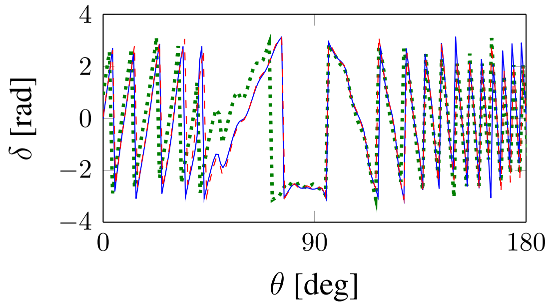


(d) $M_\infty = 0.3$, $ka = 16$

Figure 7: SPL along an arc of field points with radius $r = 8a$ at non-dimensional frequencies $ka = 4$, 16 and the problem described in Figure 6. BE physical space [23], BE transformed space (see Eq. (31)), FE reference (see Eq. (1)).



(a) $M_\infty = 0.1$, $ka = 16$



(b) $M_\infty = 0.3$, $ka = 16$

Figure 8: Phase of the pressure field along an arc of field points with radius $r = 8a$ for a non dimensional frequency $ka = 16$ and the problem described in Figure 6: BE physical space [23] (solid), BE transformed space (dashed) (see Eq. (31)), FE reference (dotted) (see Eq. (1)).

7.2 Assessment of the iterative approach

The test case presented in Figure 6 is also used to assess the convergence of the iterative scheme proposed in Sec. 6.4. We use Eq. (31), i.e. the formulation where all the terms including the tangent derivatives of $\tilde{\phi}$ are retained on the l.h.s., as a reference solution $\tilde{\phi}(\mathbf{x})_{ref}$. In Figure 9 the L_2 error on the velocity potential $\tilde{\phi}$, given by

$$E_{L^2} = 100 \times \sqrt{\frac{\int_{\Gamma_{fp}} \|\tilde{\phi}(\mathbf{x}) - \tilde{\phi}(\mathbf{x})_{ref}\|^2 dS}{\int_{\Gamma_{fp}} \|\tilde{\phi}_{ref}(\mathbf{x})\|^2 dS}}, \tag{41}$$

is measured along an arc of field points with radius $r = a$ against the number of iterations N for $M_\infty = 0.1$, 0.2 and 0.3 at non-dimensional frequencies $ka = 2$, 4, 8 and 16.

The results show that the error is about $7 - 8$ % for $N = 1$, i.e. neglecting the tangential derivatives in Eq. (31), and $M_\infty = 0.3$. The error at $N = 1$ increases with M_∞ and decreases to 0.01 % within 2-3 iterations almost independently of frequency (see Figure 9). The error obtained with $N = 3$ is two orders of magnitude lower than the error given by the physical model, namely the error given using Eq. (3) in lieu of Eq. (1) [2].

8. CONCLUSIONS

A boundary integral solution in a transformed Taylor-Lorentz space-time has been proposed to solve wave propagation in weakly non-uniform subsonic potential flows. The Taylor-Lorentz transform has allowed the weakly non-uniform flow wave equation to be reduced to the standard Helmholtz formulation for which boundary conditions and source distributions in the transformed space have been presented. On this basis, a boundary element solution has been devised and an iterative approach has been proposed to allow standard boundary element kernels [18] for sound propagation in quiescent media to be also used with a mean flow.

Numerical results have shown that the solution in the transformed Taylor-Lorentz space is in good agreement with a boundary element solution provided for the weakly non-uniform flow wave equation without using the transformation [2, 23]. For $M_\infty \leq 0.3$, the proposed iterative approach has converged almost independently of frequency and Mach number with a residual error of 0.01 % within 3 iterations.

Acknowledgments

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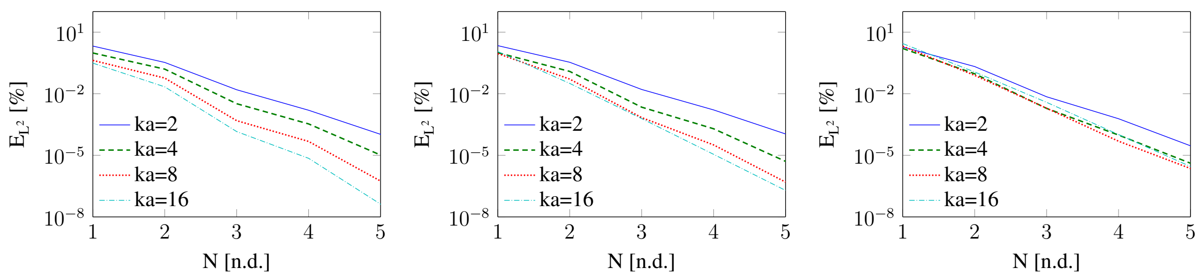

(a) $M_\infty = 0.1$ (b) $M_\infty = 0.2$ (c) $M_\infty = 0.3$

Figure 9: L_2 error measured along an arc of field point with radius $r = a$ against the number of iterations N, for the iterative approach given in Eq. (36) and the problem described in Figure 6. The reference solution is obtained using Eq. (31).

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