Calculation and Analysis of Natural Vibration Characteristics of Serpentine Belt Drive System Based on Beam Coupling Model

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Abstract

Serpentine belt drive system is widely used as engine front end accessory drive system. It’s a hybrid discrete-continuous system including rotational vibrations of pulleys and tensioner arm, and transverse vibrations of continuum belt spans. It’s easily to generate coupling vibrations and the annoying noise. Therefore it’s necessary to study natural vibration characteristics of the system. In this paper, an efficient method to evaluate the natural vibration characteristics of serpentine belt drive system is presented. The equations of motion of an actual n-pulley serpentine belt drive system are established, in which belt spans are modeled as axial moving beams with bending stiffness. Inclusion of bending stiffness results in belt-pulley coupling not captured in moving string models. The state space method is used to calculate natural frequencies and modes of the system. The method is verified by an example mentioned in a previous paper. Furthermore, influences of belt bending stiffness and belt axial speed on natural vibration characteristics and degree of coupling are investigated. The method can be popularized to solve eigenvalue problems on other hybrid systems composed by continuums and discrete bodies.

Keywords: Vibration, Belt Drive, Natural Frequency

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1. Introduction

Serpentine belt accessory drive system (SBADS) is now widely used as vehicle engine front end accessory drive system due to its advantages: compact layout, high capacity of power transfer and so on. As shown in Figure 1, the system is commonly composed of a serpentine belt, several pulleys and an automatic tensioner (including a tensioner arm, a tensioner pulley, a spring component and a damping component). The crankshaft pulley is connected to the crankshaft and rotates with the crankshaft together. Since the rotating speed of crankshaft is fluctuating, the belt spans and pulleys also vibrate. The alternating stresses caused by vibrations would accelerate the fatigue of components. Meanwhile, the annoying noise may be excited. Existing studies revealed that at a specific rotating speed, the resonance may occur in the accessory drive system. It will badly affect the NVH performance of the vehicle \cite{1}. Therefore, it’s necessary to study the natural vibration characteristics of the accessory drive system.

The major form of vibrations of serpentine belt accessory drive system includes rotational vibration of tensioner arm and pulleys, longitudinal vibration of belts in its moving direction, transverse vibration of belts in the vertical direction, lateral vibration of belts and torsional vibration of belts. Existing studies indicated that the frequency bands of longitudinal vibration and torsional vibration are much higher than the working frequency band of the engine \cite{2}. And the lateral vibration could be avoided by improving the accuracy of machining and installation of the components. As a consequence, currently only the rotational vibrations of the tensioner arm and pulleys and transverse vibrations of belt spans are considered to analyze the vibration characteristics of the SBADS.
Under the condition of ignoring the bending stiffness of the belt, some researchers established dynamic models of the SBADS. In these models, belt spans were simplified as axial moving strings. Beikmann et al. [3] simplified the belt spans as axial moving strings, and established a three pulleys-belt string coupling model, and proposed a double iterative method based on Holzer method to calculate the natural frequency of the system. Aiming at the system established by Beikmann, Zhang and Zu [4] deduced the explicit equations of the three pulleys-belt system to calculate the natural frequency. Parker [5] solved the eigenvalue problem of an actual n pulleys-belt system. He discretized the transverse vibrations of the belt spans adjacent to the tensioner arm by a series of basis function and enforced the geometric boundary conditions at the belt-tensioner interface by Lagrange multipliers. Furthermore the eigensensitivity of serpentine belt drive system was studied. Hou et al [6] proposed a new approach based on reduced characteristic determinant to obtain the natural frequency of the SBADS. This approach eliminates extra singularity of the original characteristic determinant and avoids the cases of obtaining pseudo eigenvalues or missing the real ones. Wang et al [7] proposed a hybrid algorithm to obtain the natural frequency of SBADS. It is a modified one based on the approach mentioned in [3].

When the belt spans are simplified as strings, the rotational vibrations of pulleys and tensioner arm only couple with the transverse vibrations of the two spans adjacent to the tensioner pulley. The study in [8] indicated that the transverse vibrations of all spans couple with the rotational vibrations. Therefore, the calculation results are not accurate when ignoring the belt bending stiffness. Kong and Parker et al [9] simplified the belt spans as axial moving Bernoulli-Euler beams and established a three pulleys-belt beam coupling model. They obtained the natural vibration characteristics by discretizing the motion of belt spans in space and time domain using Galerkin method. Shangguan et al [10] established a single moving beam model to obtain the natural frequency of transverse vibrations of belt spans. Their study indicated that when the bending stiffness and axial speed of the belt spans were both small, the single moving beam model were capable to obtain the natural frequency of transverse vibrations of belt spans. But up to now, beam coupling model has only been conducted to a simplified three pulleys-belt drive system. When the number of pulleys increases, the coupling of the system will be more complicated so that the equations of motion of the system will be more complicated. There is little research on an actual n pulleys-belt drive system using beam coupling model.

In this paper, an efficient method to evaluate the natural vibration characteristics of serpentine belt drive system is presented. The equations of motion of an actual n-pulley serpentine belt drive system are established in which the belt spans are modeled as axial moving beams with bending stiffness. Inclusion of bending stiffness results in belt-pulley coupling not captured in moving string models. The state space method is used to calculate natural frequencies and modes of the system. The method is verified by an example mentioned in a previous paper. Furthermore, influences of belt bending stiffness and belt axial speed on the natural vibration characteristics and degree of coupling have been investigated.

2. Mathematical formulation

Figure 2 shows a prototypical system of a serpentine belt drive including 7 pulleys: crankshaft,
air conditioner, alternator, idler, power steering, water-pump and tensioner pulley. When assuming the bending stiffness and the axial speed of the belt to be zero, the static equilibrium is defined as the reference state of the system. At the reference state, the angle of each pulley and the transverse displacement of each belt span are zero, and the belts spans are equivalent to axial moving strings. The stable state of the system is defined when the speed of the belt spans are constant and the pulleys are subjected to steady accessory torques. At the stable state, transverse vibration displacements of each span are constant.

Figure 2 – Sketch of serpentine belt accessory drive system

Key assumptions for simplification of the model are: the belt is uniform and isotropic and it stretches in a quasi-static manner; belt-pulley wedging and belt slip at the belt-pulley interfaces are neglected; pulleys are considered as rigid bodies with fixed axes and belts are considered as axial moving Bernoulli-Euler beams with bending stiffness; damping is not considered; the tensioner arm is disposed as a torsional spring with the stiffness $k_t$; belt-pulley contact points are determined at equilibrium and keep constant.

The pulleys are numbered as Figure 2. One side of the tensioner arm is fixed and the other side is connected with the center of the tensioner pulley. The belt between pulley $i$ and pulley $i+1$ is named span $i$ which length is $l_i$; $l$ denotes the average length of all spans; radius and rotary inertia of pulley $i$ are $r_i$ and $J_i$, respectively. The parameters of the tensioner arm include $r_t$, $k_t$ and $J_t$, denoting the length, spring stiffness and rotary inertia of the tensioner arm, respectively. The speed of the belt is $c$.

$\theta_i(t)$ is the rotational vibration displacement of pulley $i$ with the positive defined as the belt forward direction and $\theta_i(t)$ is the rotational vibration displacement of the tensioner arm with the positive defined clockwise. $w_i(x, t)$ is the transverse vibration deflection of span $i$ (taken positive for deflection toward the interior of the belt loop). It is a function of time $t$ and spatial coordinate $x$. It is emphasized that $w_i(x, t)$ is measured relative to the reference state corresponding to a stationary, string model system subject to any steady accessory torques. $EA$ and $EI$ are the longitudinal stiffness and bending stiffness of the belt and $m$ denotes belt mass per unit length. $\beta_1$, $\beta_2$ are the angles between the tensioner arm and the belt spans adjacent to it.

The equations of motion of the belt spans are derived using Hamilton’s principle:

$$m \left( \frac{\partial^2 w_i}{\partial t^2} - 2c \frac{\partial w_i}{\partial x} + c^2 \frac{\partial^2 w_i}{\partial x^2} \right) - \left( P_i + P_t \right) \frac{\partial^2 w_i}{\partial x^2} + EI \frac{\partial^4 w_i}{\partial x^4} = 0 \quad i=1,2,\ldots,n.$$  

The boundary conditions are:

$$w_i(0, t) = 0 \quad EI \frac{\partial^3 w_i}{\partial x^3}(0, t) = \frac{EI}{r_1} \quad w_i(l_i, t) = r_i \theta_i \cos \beta_i \quad EI \frac{\partial^4 w_i}{\partial x^4}(l_i, t) = -\frac{EI}{r_2}$$
\[ w_2(0, t) = r_i \theta_i \cos \beta_i \]
\[ EI \frac{\partial^2 w_2}{\partial x^2}(0, t) = -\frac{EI}{r_i} \]
\[ w_2(l_i, t) = 0 \]
\[ EI \frac{\partial^2 w_2}{\partial x^2}(l_i, t) = \frac{EI}{r_i} \]
\[ w_i(0, t) = w_i(l_i, t) = 0 \quad i = 3, 4, \ldots, n. \]
\[ EI \frac{\partial^2 w_i}{\partial x^2}(0, t) = \pm \frac{EI}{r_i} \quad EI \frac{\partial^2 w_i}{\partial x^2}(l_i, t) = \pm \frac{EI}{r_{i+1}} \quad i = 3, 4, \ldots, n; \text{ when } i + 1 > n, i + 1 = 1. \]

For pulley \( i \), if it works by the wedge surface, the corresponding boundary conditions defined in Eq. (4) are positive. Otherwise, if it works by the flat surface, they are negative.

The equation of motion of the tensioner arm is \[ \dot{\theta}_i + k_i \dot{\theta}_i + \left[ mc \frac{\partial w_i}{\partial t}(l_i) + (P_i - mc^2 + \ddot{P}_i) \right] \frac{\partial w_i}{\partial x}(l_i) - EI \frac{\partial^3 w_i}{\partial x^3}(l_i)|_r \cos \beta_i + (mc^2 - P_i) \sin \beta_i \]
\[ - \left[ mc \frac{\partial w_i}{\partial t}(0) + (P_i - mc^2 + \ddot{P}_i) \frac{\partial w_i}{\partial x}(0) - EI \frac{\partial^3 w_i}{\partial x^3}(0)|_r \cos \beta_i + (mc^2 - P_i) \sin \beta_i = 0 \]

\( P_i (i = 1, 2, \ldots, n) \) denotes the belt tension of span \( i \) under reference state. \( \ddot{P}_i \) is the dynamic belt tension when the belt moves with a constant speed \( c \). They can be obtained as follows:

\[ \ddot{P}_i = \frac{EA}{l_i} \left\{ -r_i \theta_i + r_i \dot{\theta}_i - r_i \theta_i \sin \beta_i + \int_0^l \left( \frac{\partial w_i}{\partial x} \right)^2 dx \right\} \]
\[ \ddot{P}_i = \frac{EA}{l_i} \left\{ -r_i \dot{\theta}_i - r_i \dot{\theta}_i - r_i \theta_i \sin \beta_i + \int_0^l \left( \frac{\partial w_i}{\partial x} \right)^2 dx \right\} \]
\[ \ddot{P}_i = \frac{EA}{l_i} \left\{ -r_i \theta_i \dot{\theta}_i + r_i \theta_i \dot{\theta}_i + \int_0^l \left( \frac{\partial w_i}{\partial x} \right)^2 dx \right\} \]

The equations of motion of pulleys are:

\[ J_i \ddot{\theta}_i - \ddot{P}_i \cos \beta_i + \ddot{P}_i \sin \beta_i = M_i \quad i = 1, 2, \ldots, n; \text{ when } i - 1 < i, i - 1 = n. \]

In Eq. (9) \( \dot{M}_i \) are dynamic torques of each pulley, which denotes the torque change relative to the reference state.

For facilitating derivation and calculation, the following non-dimensional variables are defined:

\[ \hat{x}_i = \frac{x_i}{l_i}, \quad \hat{w}_i = \frac{w_i}{w_0}, \quad \hat{t} = \frac{t}{T_0}, \quad \hat{P}_i = \frac{P_i}{P_0}, \quad \ddot{\theta}_i = \frac{\ddot{\theta}_i}{\omega_i^2}, \quad s = c \frac{m}{m_0} \]
\[ \hat{k}_i = \frac{k_i}{P_0^2}, \quad \gamma = \frac{EA}{P_0^2}, \quad \hat{m}_i = \frac{m_i}{m_0}, \quad \hat{m}_i = \frac{m_i}{m_0}, \quad \hat{\theta}_i = \frac{\theta_i}{\omega_i}, \quad \hat{M}_i = \frac{M_i}{P_0 r_i} \]

In Eq.(10), \( P_0 \) is the initial tension at the reference state with no accessory torques.

Eqs.(1-9) describe the steady-state configuration of the system completely. Linearization for small motions about the steady-state configuration of the system leads to:

\[ w_i = w_i^* + \delta w_i, \quad \theta_i = \theta_i^* + \delta \theta_i, \quad \theta_i = \theta_i^* + \delta \theta_i \]
\[ \frac{\partial w_i}{\partial x^2} = \frac{\partial^3 w_i}{\partial x^3} + \frac{\partial^3 w_i}{\partial x^3}, \quad \frac{\partial^3 w_i}{\partial x^3} = \frac{\partial^3 w_i}{\partial x^3} \]
\[ \frac{\partial \theta_i}{\partial t} = \frac{\partial \theta_i}{\partial t} \quad \frac{\partial \theta_i}{\partial t} = \frac{\partial \theta_i}{\partial t} \]

In Eqs.(11-12), the quantities denoted by asterisks are the steady motions of each component and the quantities denoted by “\( \delta \)” are the small vibrations about the steady-state configuration. Substituted Eqs.(10-12) into Eqs.(1-9) leads to the dynamic equations of the system relative to the steady-state configuration. It is emphasized that \( w_i(x, t), \theta_i(t) \) and \( \theta_i(t) \) now represent small vibration about the steady motion (not about the reference state as in Eqs.(1-9)). For convenience, the hats on dimensionless variables are dropped.

The span vibration equations are:
\[ \left( \frac{l_1}{l} \right) \frac{\partial^2 w_1}{\partial t^2} - 2s \left( \frac{l_1}{l} \right) \frac{\partial^2 w_1}{\partial x \partial t} - \frac{P}{l_1} \frac{\partial^2 w_1}{\partial x^2} + \varepsilon \left( \frac{l_1}{l} \right)^2 \frac{\partial^4 w_1}{\partial x^4} = 0 \]  

\[ w_1(0,t) = 0 \quad \frac{\partial^2 w_1}{\partial x^2}(0,t) = 0 \quad w_1(1,t) = \frac{r_1 \theta_1 \cos \beta_1}{l_1} \quad \frac{\partial^2 w_1}{\partial x^2}(1,t) = 0 \]  

\[ \left( \frac{l_2}{l} \right)^2 \frac{\partial^2 w_2}{\partial t^2} - 2s \left( \frac{l_2}{l} \right) \frac{\partial^2 w_2}{\partial x \partial t} - \frac{P}{l_2} \frac{\partial^2 w_2}{\partial x^2} + \varepsilon \left( \frac{l_2}{l} \right)^2 \frac{\partial^4 w_2}{\partial x^4} - \]  

\[ \frac{\gamma}{l_2} \left[ r_2 \theta_2 - r_2 \theta_2 - r_2 \theta_2 \sin \beta_2 + l_2 \int_0^1 \frac{\partial w^*_2}{\partial x} \frac{\partial w_2}{\partial x} \, dx \right] \frac{\partial^2 w_2^*}{\partial x^2} = 0 \]  

\[ w_2(0,t) = \frac{r_2 \theta_2 \cos \beta_2}{l_2} \quad \frac{\partial^2 w_2}{\partial x^2}(0,t) = 0 \quad w_2(1,t) = 0 \quad \frac{\partial^2 w_2}{\partial x^2}(1,t) = 0 \]  

\[ \left( \frac{l_1}{l} \right) \frac{\partial^2 w_1}{\partial t^2} - 2s \left( \frac{l_1}{l} \right) \frac{\partial^2 w_1}{\partial x \partial t} - \frac{P}{l_1} \frac{\partial^2 w_1}{\partial x^2} + \varepsilon \left( \frac{l_1}{l} \right)^2 \frac{\partial^4 w_1}{\partial x^4} - \]  

\[ \frac{\gamma}{l_1} \left[ r_1 \theta_1 - r_1 \theta_1 - r_1 \theta_1 \sin \beta_1 + l_1 \int_0^1 \frac{\partial w^*_1}{\partial x} \frac{\partial w_1}{\partial x} \, dx \right] \frac{\partial^2 w_1^*}{\partial x^2} = 0 \]  

\[ w_1(0,t) = 0 \quad w_1(1,t) = 0 \quad \frac{\partial^2 w_1}{\partial x^2}(0,t) = 0 \quad \frac{\partial^2 w_1}{\partial x^2}(1,t) = 0 \]  

\[ P_i^* = \frac{\gamma}{l_i} \left[ -r_i \theta_{i+1}^* + r_i \theta_i^* \sin \beta_i + l_i \int_0^1 \frac{\partial w^*_i}{\partial x} \frac{\partial w_i}{\partial x} \, dx \right] \quad i=3,4,...n; \text{ when } i+1>n, i+1=1. \]  

\[ P_i^* = \frac{\gamma}{l_i} \left[ -r_i \theta_{i+1}^* + r_i \theta_i^* \sin \beta_i + l_i \int_0^1 \frac{\partial w^*_i}{\partial x} \frac{\partial w_i}{\partial x} \, dx \right] \quad i=3,4,...n. \]  

\[ P_i^* = \frac{\gamma}{l_i} \left[ -r_i \theta_{i+1}^* + r_i \theta_i^* + l_i \int_0^1 \frac{\partial w^*_i}{\partial x} \frac{\partial w_i}{\partial x} \, dx \right] \quad i=3,4,...n; \text{ when } i+1>n, i+1=1. \]  

The pulleys and tensioner arm equations are:

\[ m_1 \frac{\partial^2 \theta_1}{\partial t^2} + \frac{\gamma}{l_1} \left[ r_1 \theta_1 - r_1 \theta_2 - r_1 \theta_1 \sin \beta_1 + l_1 \int_0^1 \frac{\partial w^*_1}{\partial x} \frac{\partial w_1}{\partial x} \, dx \right] \frac{\partial^2 \theta_1}{\partial x^2} = 0 \]  

\[ m_2 \frac{\partial^2 \theta_2}{\partial t^2} + \frac{\gamma}{l_2} \left[ r_2 \theta_2 - r_2 \theta_1 + r_2 \theta_2 \sin \beta_2 + l_2 \int_0^1 \frac{\partial w^*_2}{\partial x} \frac{\partial w_2}{\partial x} \, dx \right] \frac{\partial^2 \theta_2}{\partial x^2} = 0 \]  

\[ m_3 \frac{\partial^2 \theta_3}{\partial t^2} + \frac{\gamma}{l_3} \left[ r_3 \theta_3 - r_3 \theta_1 + l_3 \int_0^1 \frac{\partial w^*_3}{\partial x} \frac{\partial w_3}{\partial x} \, dx \right] \frac{\partial^2 \theta_3}{\partial x^2} = 0 \]  

\[ m_4 \frac{\partial^2 \theta_4}{\partial t^2} + \frac{\gamma}{l_4} \left[ r_4 \theta_4 - r_4 \theta_1 + l_4 \int_0^1 \frac{\partial w^*_4}{\partial x} \frac{\partial w_4}{\partial x} \, dx \right] \frac{\partial^2 \theta_4}{\partial x^2} = 0 \]  

\[ m_5 \frac{\partial^2 \theta_5}{\partial t^2} + \frac{\gamma}{l_5} \left[ r_5 \theta_5 - r_5 \theta_1 + l_5 \int_0^1 \frac{\partial w^*_5}{\partial x} \frac{\partial w_5}{\partial x} \, dx \right] \frac{\partial^2 \theta_5}{\partial x^2} = 0 \]  

\[ m_i \frac{\partial^2 \theta_i}{\partial t^2} + \frac{\gamma}{l_i} \left[ r_i \theta_i - r_i \theta_{i-1} + l_i \int_0^1 \frac{\partial w^*_i}{\partial x} \frac{\partial w_i}{\partial x} \, dx \right] \frac{\partial^2 \theta_i}{\partial x^2} = 0 \]  

\[ m_i \frac{\partial^2 \theta_i}{\partial t^2} + \frac{\gamma}{l_i} \left[ r_i \theta_i - r_i \theta_{i-1} + l_i \int_0^1 \frac{\partial w^*_i}{\partial x} \frac{\partial w_i}{\partial x} \, dx \right] \frac{\partial^2 \theta_i}{\partial x^2} = 0 \]  

\[ m_i \frac{\partial^2 \theta_i}{\partial t^2} + \frac{\gamma}{l_i} \left[ r_i \theta_i - r_i \theta_{i-1} + l_i \int_0^1 \frac{\partial w^*_i}{\partial x} \frac{\partial w_i}{\partial x} \, dx \right] \frac{\partial^2 \theta_i}{\partial x^2} = 0 \]  

\[ m_i \frac{\partial^2 \theta_i}{\partial t^2} + \frac{\gamma}{l_i} \left[ r_i \theta_i - r_i \theta_{i-1} + l_i \int_0^1 \frac{\partial w^*_i}{\partial x} \frac{\partial w_i}{\partial x} \, dx \right] \frac{\partial^2 \theta_i}{\partial x^2} = 0 \]  

\[ i=4,5,...n. \]
In Eqs. (13-27), \( w_i^*, \theta_i^*, \theta_i^*, P_i^* \) denote the steady motion quantities of the system, which can be obtained using boundary value problem method mentioned in [8]. Through Eqs. (13), (15) and (17), it is obvious that transverse vibrations of all the belt spans couple with the rotational vibrations of the pulleys. The degree of pulley-belt vibration coupling is determined mainly by the equilibrium curvature \( \frac{\partial^2 w_i^*}{\partial x^2} \) of each span. A parameter can be defined as the coupling indicator for each span as follows [8]:

\[
\Gamma_i = \int_0^L \left( \frac{\partial^2 w_i^*}{\partial x^2} \right)^2 dx
\]

(28)

And the coupling indicator for the whole system is:

\[
\Gamma = \sum_{i=1}^n \Gamma_i
\]

(29)

When the coupling between the rotation motions and transverse motions increases, the coupling indicator \( \Gamma \) increases.

Eqs. (14) and (16) show that the boundary conditions of span 1 and span 2 couple with the ones of the tensioner arm. To decouple the boundary conditions, the following coordinating transformation is conducted:

\[
y_i = w_i - \frac{L}{l_1} x \cos \beta_i \theta_i \quad y_2 = w_2 + \frac{L}{l_2} (x-1) \cos \beta_2 \theta_i \quad y_i = w_i \quad i=3,4,\ldots,n.
\]

(30)

Substituted Eq. (30) into (14), (16) and (18), one can get the following boundary conditions:

\[
y_i (0,t) = y_i (1,t) = 0 \quad \frac{\partial^2 y_i (0,t)}{\partial t^2} = \frac{\partial^2 y_i (1,t)}{\partial t^2} = 0 \quad i=1,2,\ldots,n.
\]

(31)

Serpentine belt drive system is a typical hybrid one including continuums and discrete bodies. The motions of pulleys and tensioner arm are discrete but the motions of belt spans are continuous in space. Therefore, it is necessary to discretize the transverse deflections of belt spans. The classical Galerkin method is used here. According to the boundary conditions in Eq. (31), the sinusoidal function is chosen as the basic function. So the transverse deflections of belt spans can be assumed as:

\[
y_i (x,t) = \sum_{k=1}^{N_i} a_{ik} (t) \sin k \pi x \quad i=1,2,\ldots,n.
\]

(32)

where \( N_i \) denotes the number of basic functions, \( a_{ik} (t) \) is a function of time and can be expressed as \( a_{ik} (t) = b_{ik} e^{i\omega t} \).

Utilizing the orthogonality of basic functions, the equations of motion of the system can be written in matrix form where \( a_{ik} (t) \), \( \theta_i (t) \) and \( \theta_i^* (t) \) are variables. In order to maintain the symmetry of the mass and stiffness matrices, equations of the belt spans are multiplied by the factor \( l_i/l_i \) and those of the pulleys are multiplied by the factor \( r_i/r_i^* \). The equation of the tensioner arm is transformed by the approach mentioned in [9]. After that, the equations of motion of the system can be written as:

\[
M \ddot{Y} + C \dot{Y} + K Y = F^0
\]

(33)
where $Y = \{a_{i1}, a_{i2}, \ldots, a_{iN_i}, \ldots, a_{i1}, \ldots, a_{i1}, \theta_{i1}, \ldots, \theta_{i1}\}^T$, $F_0 = [0, 0, \ldots, 0, M_{r_1}/r_1, \ldots, M_{r_n}/r_n, 0]^T$, $p = \sum_{i=1}^{N_i}$. $M^0, K^0$ are both symmetric and $C^0$ is skew-symmetric, as follows.

$$
M^0 = \begin{bmatrix}
M_{1,1} & 0 & 0 & 0 & 0 & 0 & M_{1,2n+1} \\
M_{2,2} & 0 & 0 & 0 & 0 & 0 & M_{2,2n+1} \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
M_{n,n} & 0 & 0 & 0 & 0 & 0 & M_{n,2n+1} \\
& & \frac{r_m_i}{r_i} & 0 & 0 & 0 \\
& & & \frac{r_m_i}{r_i} & 0 & 0 \\
& & & & \text{sym} & \ddots & \ddots \\
& & & & & \frac{r_m_i}{r_i} & 0 \\
& & & & & & \frac{r_m_i}{r_i} \\
\end{bmatrix}
$$

(34)

$$
C^0 = \begin{bmatrix}
C_{1,1} & C_{1,2n+1} \\
C_{2,2} & C_{2,2n+1} \\
& \ddots & \ddots & \ddots \\
C_{n,n} & 0 & 0 \\
& & \text{sym} & \ddots \\
& & & \text{sym} \\
\end{bmatrix}
$$

(35)

$$
K^0 = \begin{bmatrix}
K_{1,1} & K_{1,n+1} & K_{1,n+2} & \ldots & K_{1,2n+1} \\
K_{2,2} & K_{2,n+1} & K_{2,n+2} & \ldots & K_{2,2n+1} \\
& \ldots & \ldots & \ldots & \ldots \\
K_{n,n} & K_{n,n+1} & K_{n,n+2} & \ldots & K_{n,2n+1} \\
& & \ldots & \ldots & \ldots \\
& & & \text{sym} \\
& & & & \text{sym} \\
\end{bmatrix}
$$

(36)

3. Solution of eigenvalue problem

For an actual serpentine belt drive, vibrations of the system are induced by the rotational fluctuation of the crankshaft. So the angle displacement of crankshaft pulley $\theta_i$ should be considered as the excitation. It leads to the equations of free vibration of the system:

$$M\ddot{\mathbf{D}} + C\dot{\mathbf{D}} + K\mathbf{D} = 0$$

(37)

where $D = \{a_{i1}, a_{i2}, \ldots, a_{iN_i}, \ldots, a_{i1}, \theta_{i1}, \ldots, \theta_{i1}\}^T$; $M, C, K$ are the new matrices transformed by $M^0, C^0, K^0$ respectively with the row $p+1$ and column $p+1$ removed.

Introducing a state space variable $\mathbf{Z}$, Eq.(37) can be written as:

$$M^*\ddot{\mathbf{Z}} + K^*\mathbf{Z} = 0$$

(38)
where \( Z = \begin{bmatrix} D & \dot{D} \\ \dot{D}^T & 0 \end{bmatrix}, M^* = \begin{bmatrix} 0 & M \\ M^T & 0 \end{bmatrix}, K^* = \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix}. \)

Assuming that \( Z = z e^{\lambda t} \), it leads to the eigenvalue problem:

\[
(K^* - \omega^2 M^*)z = 0
\]  

(39)

Natural frequencies of the system \( \omega_i (i=1,2,\ldots,2n) \) and corresponding eigenvectors \( \varphi_i (i=1,2,\ldots,2n) \) can be obtained by \( |K^* - \omega^2 M^*| = 0 \). Noticing that \( \omega_i \) are dimensionless natural angular frequency. It can be transformed into natural frequencies \( f_i (\text{Hz}) \) by:

\[
f_i = \frac{\omega_i}{2\pi} \sqrt{\frac{p_0}{m^2}}
\]  

(40)

To evaluate the dominant characteristics of the mode, modal kinetic energy percentage is introduced. When the system vibrates as the \( j \) order mode, the modal kinetic energy percentage of DOF \( k \) is \([11]\):

\[
E(k, j) = \frac{\frac{1}{2} \omega^2 \sum_{i=1}^{p+n} m_i \varphi_i \varphi_j}{\frac{1}{2} \omega^2 \sum_{i=1}^{p+n} m_i \varphi_i} = \frac{\varphi_i^T M \varphi_j}{\varphi_j^T M \varphi_j}
\]  

(41)

In Eq.(41), \( \varphi_j \) is the \( j \) order mode shape of the system and \( m_{i_l} \) is the element at row \( k \) and column \( l \) in matrix \( M \). The larger \( E(k, j) \) means that in the \( j \) order mode the vibration of DOF \( k \) is more predominant.

4. Results and discussion

4.1 Verification of the method

Firstly, the calculation method should be verified. A prototype of a 7 pulleys-belt system shown in Figure 2 and analyzed in [5] is calculated. The results obtained by the method in this paper and are contrasted with the ones shown in [5]. The rotating speed of crankshaft is set to \( n=680 \text{rpm} \). The parameters of the system are listed in Tables 1–4.

<table>
<thead>
<tr>
<th>Table 1 – Parameters of the belt</th>
</tr>
</thead>
<tbody>
<tr>
<td>EA(N)</td>
</tr>
<tr>
<td>111200</td>
</tr>
</tbody>
</table>

Note: ⎯ ⎯ means this parameter is a variable, it will be explained as follows

<table>
<thead>
<tr>
<th>Table 2 – Parameters of the tensioner</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_1 )(mm)</td>
</tr>
<tr>
<td>79.9</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 3 – Parameters of the pulleys</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pulley number</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>6</td>
</tr>
<tr>
<td>7</td>
</tr>
</tbody>
</table>
Table 4 – Parameters of the belt spans

<table>
<thead>
<tr>
<th>Span number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_i$(N)</td>
<td>257</td>
<td>257</td>
<td>285.9</td>
<td>568.7</td>
<td>568.7</td>
<td>881.1</td>
<td>1278</td>
</tr>
<tr>
<td>$l_i$(mm)</td>
<td>92.71</td>
<td>114.4</td>
<td>169.2</td>
<td>267.7</td>
<td>136.6</td>
<td>178.8</td>
<td>209.0</td>
</tr>
</tbody>
</table>

Reference [5] showed the first 10 order natural frequencies calculated by string coupling model. When using string coupling model, the rotational vibrations of pulleys and tensioner arm only couple with the transverse vibrations of the two spans adjacent to the tensioner pulley. As a result, natural frequencies of other spans could not be obtained. The smaller the bending stiffness is, the moving belt is more approximate to string. Assuming a small bending stiffness $\varepsilon = 0.02$, the natural frequencies are obtained by the method presented in this paper and contrasted with the results in reference [5]. The rotational speed of the crankshaft is 680rpm. The results are shown in Table 5.

Table 5 – Natural Frequencies of the System

<table>
<thead>
<tr>
<th>Order</th>
<th>Results in this paper(Hz)</th>
<th>Results in <a href="Hz">5</a></th>
<th>Mode type</th>
<th>Relative error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>32.80</td>
<td>32.85</td>
<td>1st rotational mode</td>
<td>0.148</td>
</tr>
<tr>
<td>2</td>
<td>79.37</td>
<td>79.48</td>
<td>2nd rotational mode</td>
<td>0.131</td>
</tr>
<tr>
<td>3</td>
<td>135.58</td>
<td>—</td>
<td>Span 4 1st transverse mode</td>
<td>—</td>
</tr>
<tr>
<td>4</td>
<td>151.63</td>
<td>—</td>
<td>Span 3 1st transverse mode</td>
<td>—</td>
</tr>
<tr>
<td>5</td>
<td>178.72</td>
<td>178.7</td>
<td>3rd rotational mode</td>
<td>0.016</td>
</tr>
<tr>
<td>6</td>
<td>212.87</td>
<td>210.0</td>
<td>Span 2 1st transverse mode</td>
<td>1.371</td>
</tr>
<tr>
<td>7</td>
<td>253.14</td>
<td>—</td>
<td>Span 6 1st transverse mode</td>
<td>—</td>
</tr>
<tr>
<td>8</td>
<td>260.98</td>
<td>—</td>
<td>Span 7 1st transverse mode</td>
<td>—</td>
</tr>
<tr>
<td>9</td>
<td>263.47</td>
<td>258.9</td>
<td>Span 1 1st transverse mode</td>
<td>1.767</td>
</tr>
<tr>
<td>10</td>
<td>265.98</td>
<td>—</td>
<td>Span 5 1st transverse mode</td>
<td>—</td>
</tr>
<tr>
<td>11</td>
<td>271.49</td>
<td>—</td>
<td>Span 4 2nd transverse mode</td>
<td>—</td>
</tr>
<tr>
<td>12</td>
<td>288.97</td>
<td>292.0</td>
<td>4th rotational mode</td>
<td>1.038</td>
</tr>
<tr>
<td>13</td>
<td>305.02</td>
<td>—</td>
<td>Span 3 2nd transverse mode</td>
<td>—</td>
</tr>
<tr>
<td>14</td>
<td>388.87</td>
<td>389.9</td>
<td>5th rotational mode</td>
<td>0.263</td>
</tr>
<tr>
<td>15</td>
<td>408.15</td>
<td>—</td>
<td>Span 4 3rd transverse mode</td>
<td>—</td>
</tr>
<tr>
<td>16</td>
<td>433.41</td>
<td>420.0</td>
<td>Span 2 2nd transverse mode</td>
<td>3.194</td>
</tr>
<tr>
<td>17</td>
<td>462.28</td>
<td>—</td>
<td>Span 3 3rd transverse mode</td>
<td>—</td>
</tr>
<tr>
<td>18</td>
<td>507.07</td>
<td>—</td>
<td>Span 6 2nd transverse mode</td>
<td>—</td>
</tr>
<tr>
<td>19</td>
<td>522.38</td>
<td>—</td>
<td>Span 7 2nd transverse mode</td>
<td>—</td>
</tr>
<tr>
<td>20</td>
<td>532.71</td>
<td>518.2</td>
<td>Span 1 2nd transverse mode</td>
<td>2.801</td>
</tr>
<tr>
<td>21</td>
<td>538.83</td>
<td>—</td>
<td>Span 5 2nd transverse mode</td>
<td>—</td>
</tr>
<tr>
<td>22</td>
<td>542.26</td>
<td>541.0</td>
<td>6th rotational mode</td>
<td>0.235</td>
</tr>
</tbody>
</table>

Note: — — means it can’t be contrasted because it wasn’t shown in [5]

It shows that the relative error between the results in this paper and those in reference [5] is very small, especially the results of rotationally dominant mode. It indicates that the calculation method in this paper is feasible. Furthermore, beam coupling model is more close to the reality. It can describe the vibration coupling of every span. The natural frequencies of all the components in SBADS can be obtained by the model in this paper, while these can’t be achieved using the string coupling model.
4.2 Influence of bending stiffness on natural frequencies

The bending stiffness of serpentine belt is mainly influenced by the number of ribs. It can be estimated by $EI = (m - 1) \times 2.867 \times 10^{-3} \text{N} \cdot \text{m}^2$, where $m$ denotes the number of ribs of the belt. Dimensionless bending stiffness $\varepsilon$ is defined in Eq.(10). For an actual SBADS, the range of $\varepsilon$ is $0.01 \leq \varepsilon \leq 0.12$. The natural frequencies are calculated under different bending stiffness and the results are shown in Figures 3-4.

Figure 3 - Natural frequency spectrum of rotationally dominant modes for varying belt bending stiffness

Figure 4 - Natural frequency spectrum of Span 1 transversely dominant modes for varying belt bending stiffness

Figure 3 indicates that the first three orders natural frequencies of rotationally dominant modes are slightly influenced by belt bending stiffness and the 4th and 5th orders are greatly influenced. With the increase of the bending stiffness, the 4th and 5th orders natural frequencies decrease clearly.

Figure 4 indicates that the natural frequencies of transversely dominant modes of belt spans are greatly influenced by belt bending stiffness. With the increase of the bending stiffness, the first five orders natural frequencies of span 1 all increase obviously. The results of other spans are the same.

Figure 5 is the coupling indicator spectrum for varying belt bending stiffness. It indicates that with the increase of the bending stiffness, the degree of coupling of the system increase. It’s consistent with the actual situation. The degree of coupling is low when bending stiffness is small, so that the belt spans can be simplified as moving strings. But the belt used in SBADS commonly has 6 ribs. The coupling becomes strong when bending stiffness increases.
4.3 Influence of belt speed on natural frequencies

The belt speed changes with the rotating speed of the engine. It’s necessary to analyze the influence of belt speed on natural frequencies. Natural frequencies are calculated when the rotating speed of the crankshaft changes from 0 rpm to 5000 rpm. Here the bending stiffness is $\varepsilon = 0.01$.

Figures 6–7 indicate that belt speed has no influence on natural frequencies of rotationally dominant modes but has a significant influence on natural frequencies of transversely dominant modes. Figure 7 indicates that the first three orders natural frequencies of transversely dominant modes of span 1 slightly decrease with the increase of the belt speed, but the 4th and 5th orders increase significantly. Other spans show the same phenomenon when the belt speed changes.
5. Conclusions

In serpentine belt drive system, coupling between rotational vibrations of pulleys and transverse vibrations of belt spans may be induced by belt bending stiffness. The equations of motion for an actual n pulley-belt system are derived in which the belt spans are simplified as moving beams with bending stiffness. The natural frequencies and mode shapes of the system are obtained by the state space method. The calculation method is verified and some parameter studies are conducted. The major conclusions include:

1) The method presented in this paper can efficiently solve the eigenvalue problem of an actual n pulley-belt drive system including an automatic tensioner.
2) The vibration modes of the system are quite complicated. They can be divided approximately into two categories: rotationally dominant modes and transversely dominant modes. Different mode types show different variations when system parameters change.
3) With the increase of the belt bending stiffness, the coupling between rotational and transverse vibrations increases significantly. The first three orders natural frequencies of rotationally dominant modes are slightly influenced by bending stiffness but the 4th and 5th orders are greatly influenced. As for natural frequencies of transversely dominant modes, they increase significantly with the increase of the belt bending stiffness. While, belt speed has no influence on natural frequencies of rotationally dominant modes but has a significant influence on those of transversely dominant modes.

Resonance may occur when the natural frequency approaches to the fluctuation frequency of the crankshaft. It can badly affect the stability of the system and induce the annoying noise. It’s necessary to avoid resonance by adjusting system parameters. Therefore, the sensitivity of natural frequency to the key parameters of the system should be studied in future work.

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