Coupling of BEM with analytic solution for shell elements in 2.5D

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ABSTRACT

Our simulation of vibrations from railway tunnels in a horizontally layered anisotropic half space is based on the boundary element method (BEM) in 2.5D for the soil and an analytic solution for a plane shell element in 2.5D for the tunnel shell. The coupling of the boundary stresses is done with the loads exiting the shell elements. A constant stress distribution is assumed for the shell elements and therefore constant shape functions are used in the BEM for the stresses. The displacements of the soil at the tunnel boundary and the shell are coupled at the nodes. Here, linear shape functions in the BEM are used. For the shell elements analytic complex valued exponential functions occur. One difficulty is that in the singularities in the stresses and displacements of the test function at the boundary occur. To be able to solve the singularities in the boundary integrals the integration over each boundary element is done in the wave number domain with respect to the horizontal direction of the cross section of the tunnel and the inverse Fourier transformation of the integrals over the elements is done numerically.

Keywords: Boundary Element Method, 75.5, Simulation of Rail Traffic Noise, 76.1.2

1. INTRODUCTION

Rail traffic produces vibrations that can be perceived directly or as secondary airborne noise in houses near the tunnel (3,6,11). Usual models do not take the layering of the soil into account (1,4,7,9,11,13,15,18,19,20,21,22,33,34). In this study a method to simulate vibrations at the free surface of a horizontally layered anisotropic linear elastic medium is presented c.f. (2,5,11,14,16,23). The layering is especially important for receiver positions far away from the tunnel.

In a first step, the homogenous soil was modelled using an inverse Fourier transformation of the horizontal directions. The inverse transformation was done numerically. In a second step, a boundary element simulation was added assuming that the tunnel shell is a hole in the stratified medium. The new approach described here allows adding a tunnel shell and an ongoing superstructure to the BEM model for the soil. It is assumed that the tunnel is straight and the superstructure is invariant in the longitudinal direction x. This allows to apply a 2.5D approach in which the longitudinal axis x is kept in the wave number domain with respect to k_x. The coupling of the BEM part with the analytic part for the tunnel shell is done in the wave number domain with respect to the longitudinal direction x. As a last step the inverse Fourier transformation of the longitudinal direction x is done numerically for the receiver positions.

2. BEM AND FEM IN 2.5D

The 2.5D simulations can simply be derived using a modification of Planscherel's theorem. The advantage of the 2.5D approach is that instead one large 3D problem a sequence of independent 2D problems has to be solved. The original theorem gives a relation between original and Fourier transformed domain:

$$\hat{f}(k_x) = \int_{-\infty}^{\infty} f(x) e^{-jk_x x} dx \quad , \quad \hat{g}(k_x) = \int_{-\infty}^{\infty} g(x) e^{-jk_x x} dx$$
$$\int_{-\infty}^{\infty} f(x)\bar{g}(x)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k_x)\bar{\hat{g}}(k_x)dk_x \ . \tag{1}$$

Please note that the bar denotes the conjugate complex of the function. Taking Hermite's condition of the Fourier transformation into account a modified version can be derived, that does not depend on conjugate complex values, but leads to a coupling of the positive and negative wavenumbers:

$$\int_{-\infty}^{\infty} f(x)g(x)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k_x)\hat{g}(-k_x)dk_x \ . \tag{2}$$

This relationship called modified Plancherel's theorem can be used to transform the boundary element method (BEM) and the finite element method (FEM) from original domain to the 2.5 D domain. Therefore, a Fourier transformation from the coordinate x to the wavenumber k_x is needed.

The BEM in 3D can be characterized by the integral with respect to the cross section Γ and the longitudinal coordinate x. A star is used, where the Green's function is applied.

$$\begin{aligned} \mathbf{c}^T\mathbf{u}(x_o,y_o,z_o) = \int_{-\infty}^{\infty}\int_{\Gamma} \mathbf{u}^T(x,y,z)\mathbf{t}^*(x,y,z,x_o,y_o,z_o)d\Gamma dx - \\ -\int_{-\infty}^{\infty}\int_{\Gamma} \mathbf{t}^T(x,y,z)\mathbf{u}^*(x,y,z,x_o,y_o,z_o)d\Gamma dx \end{aligned} \tag{3}$$

In this equation, **c** is a constant vector depending on the surface Γ. The vectors **u** are the displacements and **t** are the tractions at the boundary. (x,y,z) is a point at the boundary and (x_0,y_0,z_0) is an evaluation point (9,17).

At the left side of the equation **u** is substituted by the inverse Fourier integral transform of the related spectrum while the integrals on the right side are substituted using modified Plancherel's theorem. To simplify the equation $x_0=0$ is assumed. Other assumption lead to the same result after the evaluation of the Green's functions. As a last step, the integral equation with respect to the wavenumbers k_x is decoupled. The Green's functions are evaluated at $-k_x$ to derive equations for the unknowns at $+k_x$.

$$\begin{aligned} \mathbf{c}^T\hat{\mathbf{u}}(k_x,y_o,z_o) = \int_{\Gamma} \hat{\mathbf{u}}^T(k_x,y,z)\hat{\mathbf{t}}^*(-k_x,y,z,y_o,z_o)d\Gamma - \\ -\int_{\Gamma} \hat{\mathbf{t}}^T(k_x,y,z)\hat{\mathbf{u}}^*(-k_x,y,z,y_o,z_o)d\Gamma \end{aligned} \tag{4}$$

The derivation for the FEM equations is done in the same manner. The potential energy of the system is derived in the original 3D coordinates (x,y,z) and then transformed into 2.5D coordinates(k_x,y,z). The cross section of the domain for which the FEM is applied is denoted by Ω.

$$\Pi = \int_{-\infty}^{\infty}\int_{\Omega} \begin{bmatrix} \frac{1}{2}\boldsymbol{\varepsilon}^T(x,y,z)\mathbf{E}\boldsymbol{\varepsilon}(x,y,z) + \\ +\frac{1}{2}\omega^2\mathbf{u}^T(x,y,z)\boldsymbol{\rho}\mathbf{u}(x,y,z) + \\ +\mathbf{p}^T(x,y,z)\mathbf{u}(x,y,z) \end{bmatrix} d\Omega dx \tag{5}$$

In the transformed domain using modified Plancherel's theorem the potential becomes

$$\Pi = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{\Omega} \begin{bmatrix} \frac{1}{2}\hat{\boldsymbol{\varepsilon}}^T(k_x,y,z)\mathbf{E}\hat{\boldsymbol{\varepsilon}}(-k_x,y,z) + \\ +\frac{1}{2}\omega^2\hat{\mathbf{u}}^T(k_x,y,z)\boldsymbol{\rho}\hat{\mathbf{u}}(-k_x,y,z) + \\ +\hat{\mathbf{p}}^T(k_x,y,z)\hat{\mathbf{u}}(-k_x,y,z) \end{bmatrix} d\Omega dk_x \tag{6}$$

Again, the solution is decoupled with respect to the wave number k_x. The variation with respect to (y,z) has to be done with respect to $-k_x$ to derive equations for the unknowns at $+k_x$.

$$\delta\Pi(k_x)=\int_{\Omega}\begin{bmatrix}\hat{\boldsymbol{\varepsilon}}^T(k_x,y,z)\mathbf{E}\delta\hat{\boldsymbol{\varepsilon}}(-k_x,y,z)+\\ +\omega^2\hat{\mathbf{u}}^T(k_x,y,z)\boldsymbol{\rho}\delta\hat{\mathbf{u}}(-k_x,y,z)+\\ +\hat{\mathbf{p}}^T(k_x,y,z)\delta\hat{\mathbf{u}}(-k_x,y,z)\end{bmatrix}d\Omega=0 \tag{7}$$

3. HORIZONTALLY LAYERED ANISOTROPIC MEDIUM

A horizontally layered anisotropic medium with an isotropic half space as the lower boundary is assumed (23,24,25,26,29,31). Loads are only allowed at the interfaces of the layers. Therefore, additional interfaces are used in every depth of the impact of the needed Green's functions. The layers consist of anisotropic linear elastic material with the elasticity matrix **E**. The solution in every layer is derived in the fully Fourier transformed domain with respect to the time t, the horizontal axes x and y, and the vertical axis z. The angular frequency is ω and the wavenumbers are k_x, k_y and k_z. The differential relationship between displacements and stresses becomes a matrix equation in the transformed domain

$$\hat{\boldsymbol{\sigma}}=\begin{bmatrix}\hat{\sigma}_{xx}\\ \hat{\sigma}_{yy}\\ \hat{\sigma}_{zz}\\ \hat{\sigma}_{xy}\\ \hat{\sigma}_{yz}\\ \hat{\sigma}_{zx}\end{bmatrix}=\mathbf{E}\hat{\boldsymbol{\varepsilon}}=\mathbf{E}\hat{\mathbf{D}}\hat{\mathbf{u}}=\mathbf{E}\begin{bmatrix}jk_x & 0 & 0\\ 0 & jk_y & 0\\ 0 & 0 & jk_z\\ jk_y/2 & jk_x/2 & 0\\ 0 & jk_z/2 & jk_y/2\\ jk_z/2 & 0 & jk_x/2\end{bmatrix}\begin{bmatrix}\hat{u}_x\\ \hat{u}_y\\ \hat{u}_z\end{bmatrix}. \tag{8}$$

The equilibrium of the stresses in the medium gives three equations

$$\begin{bmatrix}jk_x\hat{\sigma}_{xx}+jk_y\hat{\sigma}_{xy}+jk_z\hat{\sigma}_{zx}\\ jk_x\hat{\sigma}_{xy}+jk_y\hat{\sigma}_{yy}+jk_z\hat{\sigma}_{yz}\\ jk_x\hat{\sigma}_{zx}+jk_y\hat{\sigma}_{yz}+jk_z\hat{\sigma}_{zz}\end{bmatrix}+\rho\Omega^2\begin{bmatrix}\hat{u}_x\\ \hat{u}_y\\ \hat{u}_z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}. \tag{9}$$

Using Eq. 8 the stresses are substituted by the displacements. A matrix relation for the displacements is the result. The singularities of the matrix with respect to the vertical wavenumber k_z leads to six generalized eigenvalues $k_{z,i}$ and eigenvectors $\boldsymbol{\Psi}_i$ for the displacements. The eigenvectors of the stresses are given by usage of the special differential matrix.

The generalized functions in the transformed domain are derived using the Dirac delta generalized function.

$$\hat{\mathbf{u}}(k_x,k_y,k_z,\Omega)=\sum_{i=1}^{6}\left[A_i\boldsymbol{\psi}_i(k_x,k_y,\Omega)\,\delta(k_z-k_{z,i})\right] \tag{10}$$

$$\hat{\boldsymbol{\sigma}}(k_x,k_y,k_z,\Omega)=\sum_{i=1}^{6}\left[\mathbf{E}\hat{\mathbf{D}}A_i\boldsymbol{\psi}_i(k_x,k_y,\Omega)\,\delta(k_z-k_{z,i})\right] \tag{11}$$

An inverse Fourier transformation with respect to the coordinate z leads to exponential functions and a differential matrix that depends on the eigenvalues.

$$\tilde{\mathbf{u}}(k_x,k_y,z,\Omega)=\sum_{i=1}^{6}\left[A_i\boldsymbol{\psi}_i(k_x,k_y,\Omega)\,e^{jk_{z,i}z}\right] \tag{12}$$

$$\tilde{\boldsymbol{\sigma}}(k_x,k_y,z,\Omega)=\sum_{i=1}^{6}\left[\mathbf{E}\tilde{\mathbf{D}}_iA_i\boldsymbol{\psi}_i(k_x,k_y,\Omega)\,e^{jk_{z,i}z}\right] \tag{13}$$

$$\widetilde{\mathbf{D}}_i = \begin{bmatrix} jk_x & 0 & 0 \\ 0 & jk_y & 0 \\ 0 & 0 & jk_{z,i} \\ jk_y/2 & jk_x/2 & 0 \\ 0 & jk_{z,i}/2 & jk_y/2 \\ jk_{z,i}/2 & 0 & jk_x/2 \end{bmatrix} \tag{14}$$

The solution has six unknown scaling factors A_i that belong to the six waves in the medium. To determine these unknown three equations for the compatibility of the displacements and three equations for the equilibrium of the stresses are used. For the interface k the following relations hold.

$$\begin{aligned} &\widetilde{\hat{\mathbf{u}}}_{k+1}(k_x,k_y,z=z_k,\Omega) = \widetilde{\mathbf{u}}_k(k_x,k_y,z=z_k,\Omega) \\ &\widetilde{\boldsymbol{\sigma}}_{r,k+1}(k_x,k_y,z=z_k,\Omega) = -\widetilde{\boldsymbol{\sigma}}_{r,k}(k_x,k_y,z=z_k,\Omega) + \mathbf{f}_k(k_x,k_y,\Omega)\ ,\ \widetilde{\boldsymbol{\sigma}}_r = \begin{bmatrix} \sigma_{xz} \\ \sigma_{yz} \\ \sigma_{zz} \end{bmatrix}. \end{aligned} \tag{15}$$

The external loading defining the Green's function is denoted by the force vector $\mathbf{f_k}$. At the free surface only three equations for the stresses can be derived. The isotropic half space has six waves three of them are up going waves. These three waves are set to zero to reduce the number of unknowns to the number of equations and to fulfil causality.

4. BOUNDARY ELEMENT METHOD IN 2.5D

The boundary element method for a tunnel structure in the soil is applied assuming that the cross section is infinitely long and straight in the longitudinal direction x. This coordinate is kept in the spectral domain, but an inverse Fourier transformation is needed for the horizontal coordinate of the cross-section.

A difficulty that arises is that the inverse Fourier transformation can only be done numerically and the BEM formulation leads to singularities (8). The solution derived here depends on a change of the sequence of integration along the boundary element and inverse Fourier integral transformation.

For the stresses, constant shape functions are assumed (28,32), because a constant pressure load on the structural elements shall be used to couple the FEM and BEM elements. For the displacements, linear shape functions are used for the boundary elements and analytic solutions are taken for the plane shell elements.

The collocation nodes are the midpoints of each elements. For this case the number of unknown displacement is in accordance with the number of collocation nodes, if the tunnel is a closed loop, The elements are split into sub elements at the depth of the collocation nodes (27,30). For every sub element with length l, a linear shape function is assumed, because a constant shape function is a special case of a linear one.

$$g(s) = \begin{cases} (as+b), & -l/2 \le s \le l/2 \\ 0 \quad , & otherwise \end{cases} \tag{16}$$

Here, a description, the double layer potential is derived but for the single layer potential a similar formulation holds. The sub element integral leads to the desired values of the element assuming the linear shape functions. The element angle with respect to the global coordinates is denoted as α.

$$\mathbf{I} = \int_{s=-l/2}^{l/2} \int_{k_y=-\infty}^{\infty} \sum_{i=1}^{6} \hat{\mathbf{T}}_i(-k_x,k_y,\alpha) e^{jk_{z,i}z} e^{jk_y y}(as+b)dk_y\ ds\ , \tag{17}$$

where the $\boldsymbol{T}_i$ contain the rotated parts of Eq.13 that are independent of z.

The sequence of the element integral and the inverse Fourier transformation is changed.

$$\mathbf{I} = \int_{k_y=-\infty}^{\infty} \sum_{i=1}^{6} \hat{\mathbf{T}}_i\left(-k_x, k_y, \alpha\right) \int_{s=-l/2}^{l/2} e^{jk_{z,i}z} e^{jk_y y}\left(as+b\right)ds\, dk_y \,. \tag{18}$$

A coordinate transformation from the element coordinate s to the global coordinates is needed.

$$z = s\sin(\alpha) + z_0 \quad , \quad y = s\cos(\alpha) + y_0$$

$$\mathbf{I} = \int_{k_y=-\infty}^{\infty} \sum_{i=1}^{6} \hat{\mathbf{T}}_i\left(-k_x, k_y, \alpha\right) e^{j\left(k_{z,i}z_0 + k_y y_0\right)} \int_{s=-l/2}^{l/2} e^{j\left(k_{z,i}\sin(\alpha) + k_y\cos(\alpha)\right)s}\left(as+b\right)ds\, dk_y \,. \tag{19}$$

A substitution is used to simplify the equation:

$$k_s = k_{z,i}\sin(\alpha) + k_y\cos(\alpha)$$

$$\mathbf{I} = \int_{k_y=-\infty}^{\infty} \sum_{i=1}^{6} \hat{\mathbf{T}}_i\left(k_x, k_y, \alpha\right) e^{j\left(k_{z,i}z_0 + k_y y_0\right)} \int_{s=-l/2}^{l/2} e^{jk_s s}\left(as+b\right)ds\, dk_y \,. \tag{20}$$

The inner integral is split into a constant part and a centred linear part.

$$\int_{s=-l/2}^{l/2} e^{jk_s s}\left(as+b\right)ds = \int_{s=-l/2}^{l/2} e^{jk_s s} as\, ds + \int_{s=-l/2}^{l/2} e^{jk_s s} b\, ds \,. \tag{21}$$

The solution for the constant part is given by

$$\int_{s=-l/2}^{l/2} e^{jk_s s} b\, ds = \left[\frac{b}{jk_s} e^{jk_s s}\right]_{s=-l/2}^{l/2} = \frac{b}{jk_s}\left(e^{jk_s l/2} - e^{-jk_s l/2}\right) = \frac{b}{jk_s} 2j\sin\left(k_s \frac{l}{2}\right) = \frac{2b}{k_s}\sin\left(k_s \frac{l}{2}\right). \tag{22}$$

and for the linear part the following solution is derived:

$$\int_{s=-l/2}^{l/2} e^{jk_s s} as\, ds = \left[\frac{a}{jk_s} e^{jk_s s} s\right]_{s=-l/2}^{l/2} - \int_{s=-l/2}^{l/2} \frac{a}{jk_s} e^{jk_s s} ds =$$

$$= \frac{al}{2jk_s}\left(e^{jk_s l/2} + e^{-jk_s l/2}\right) + \frac{a}{k_s^2}\left[e^{jk_s s}\right]_{s=-l/2}^{l/2} = \frac{-jal}{k_s}\cos\left(k_s \frac{l}{2}\right) + \frac{a}{k_s^2}\left(e^{jk_s l/2} - e^{-jk_s l/2}\right) \tag{23}$$

$$\int_{s=-l/2}^{l/2} e^{jk_s s} as\, ds = \frac{-jal}{k_s}\cos\left(k_s \frac{l}{2}\right) + \frac{a}{k_s^2} 2j\sin\left(k_s \frac{l}{2}\right) = j\left[\frac{2a}{k_s^2}\sin\left(k_s \frac{l}{2}\right) - \frac{al}{k_s}\cos\left(k_s \frac{l}{2}\right)\right].$$

The complete solution is given by:

$$\int_{s=-l/2}^{l/2} e^{jk_s s}\left(as+b\right)ds = \frac{2b}{k_s}\sin\left(k_s \frac{l}{2}\right) + ja\left[\frac{2}{k_s^2}\sin\left(k_s \frac{l}{2}\right) - \frac{l}{k_s}\cos\left(k_s \frac{l}{2}\right)\right]. \tag{24}$$

The solution for the linear part is the derivative of solution of the constant part. No singularity in $k_s=0$ occurs and therefore the solution is:

$$\lim_{k_s \to 0}\left[bl\,\mathrm{sinc}\left(k_s \frac{l}{2}\right) + j\frac{al}{k_s}\left[\mathrm{sinc}\left(k_s \frac{l}{2}\right) - \cos\left(k_s \frac{l}{2}\right)\right]\right] = bl \,. \tag{25}$$

It becomes apparent that the function gained from the element integral attenuates the spectrum for the stresses. With this attenuation, it is possible to do the inverse Fourier transformation by numerical means.

Finally, the single layer potential with a constant shape function (*a=0*) and the displacement factors independent $\mathbf{U}_i$ from z from Eq. 12 similar to the factors $\mathbf{T}_i$ lead to:

$$k_s = k_{z,i}\sin(\alpha) + k_y\cos(\alpha)$$
$$\mathbf{I} = \int_{k_y=-\infty}^{\infty} \sum_{i=1}^{6} \hat{\mathbf{U}}_i(k_x, k_y, \alpha)\, e^{j(k_{z,i}z_0 + k_y y_0)} \frac{2b}{k_s}\sin\left(k_s \frac{l}{2}\right). \tag{26}$$

The double layer potential gives:

$$k_s = k_{z,i}\sin(\alpha) + k_y\cos(\alpha)$$
$$\mathbf{I} = \int_{k_y=-\infty}^{\infty} \sum_{i=1}^{6} \hat{\mathbf{T}}_i(k_x, k_y, \alpha)\, e^{j(k_{z,i}z_0 + k_y y_0)} \left[\frac{2b}{k_s}\sin\left(k_s\frac{l}{2}\right) + ja\left[\frac{2}{k_s^2}\sin\left(k_s\frac{l}{2}\right) - \frac{l}{k_s}\cos\left(k_s\frac{l}{2}\right)\right]\right]. \tag{27}$$

5. ANALYTIC SOLUTION FOR A PLANE SHELL ELEMENT 2.5D

The tunnel structure shall be simulated by means of plane shell elements. In 2.5D, an analytic solution for a simple plane shell element is possible. For a plane shell element, the Kirchhoff plate model is used for the bending case and the plane shell model for shear effects in the plane.

5.1 Kirchhoff Plate

For a constant loading perpendicular to the plane, the Kirchhoff plate model is used. Assuming a mass density ρ, a plate stiffness K and a thickness d the following equation for the bending displacement w in direction z assuming a pressure p is derived:

$$K\left[\frac{\partial^4}{\partial x^4}w(x,y,t) + 2\frac{\partial^4}{\partial x^2 \partial y^2}w(x,y,t) + \frac{\partial^4}{\partial y^4}w(x,y,t)\right] + \rho d\frac{\partial^2}{\partial t^2}w(x,y,t) = p(x,y,t)$$
$$K = \frac{Ed^3(1-\nu^2)}{12}\ ,\quad \alpha = \sqrt{\frac{K}{\rho d}}\ . \tag{28}$$

The Fourier transformation with respect to time and the longitudinal coordinate x leads to

$$\frac{\partial^4}{\partial y^4}\hat{w}(k_x,y,\omega) - 2k_x^2\frac{\partial^2}{\partial y^2}\hat{w}(k_x,y,\omega) + \left[k_x^4 - \left(\frac{\omega}{\alpha}\right)^2\right]\hat{w}(k_x,y,\omega) = \frac{1}{K}\hat{p}(k_x,y,\omega). \tag{29}$$

The particular solution of the inhomogeneous partial differential equation is derived assuming an invariant behaviour of w with respect to the coordinate y.

$$\frac{\partial^4}{\partial y^4}\hat{w}(k_x,y,\omega) - 2k_x^2\frac{\partial^2}{\partial y^2}\hat{w}(k_x,y,\omega) + \left[k_x^4 - \left(\frac{\omega}{\alpha}\right)^2\right]\hat{w}(k_x,y,\omega) = \frac{1}{K}\hat{p}(k_x,\omega)$$
$$\hat{w}(k_x,y,\omega) = \hat{w}_0(k_x,\omega)$$
$$\left[k_x^4 - \left(\frac{\omega}{\alpha}\right)^2\right]\hat{w}_0(k_x,\omega) = \frac{1}{K}\hat{p}(k_x,\omega) \tag{30}$$
$$\hat{w}_0(k_x,\omega) = \frac{\hat{p}(k_x,\omega)}{K\left[k_x^4 - \left(\frac{\omega}{\alpha}\right)^2\right]}\ .$$

The homogenous solution is derived using exponential ansatz functions based on Eq. 28.

$$\hat{w}(k_x,y,\omega) = Ae^{ay}\ . \tag{31}$$

The four solutions are given by

$$a^4 - 2k_x^2 a^2 + \left[k_x^4 - \left(\frac{\omega}{\alpha} \right)^2 \right] = 0$$

$$a_{1,2}^2 = k_x^2 \pm \sqrt{4k_x^4 - 4\left[k_x^4 - \left(\frac{\omega}{\alpha} \right)^2 \right]} = k_x^2 \pm \frac{|\omega|}{\alpha} \tag{32}$$

$$a_{1,2,3,4} = \pm \sqrt{k_x^2 \pm \frac{|\omega|}{\alpha}}$$

The homogenous solution is given by

$$\widehat{w}(k_x, y, \omega) = A_1 e^{\sqrt{k_x^2 + \frac{|\omega|}{\alpha}} y} + A_2 e^{-\sqrt{k_x^2 + \frac{|\omega|}{\alpha}} y} + A_3 e^{\sqrt{k_x^2 - \frac{|\omega|}{\alpha}} y} + A_4 e^{-\sqrt{k_x^2 - \frac{|\omega|}{\alpha}} y} \tag{33}$$

The complete solution for displacements, rotations, distributed moment and distributed lateral force are

$$\widehat{w}(k_x, y, \omega) = A_1 e^{\sqrt{k_x^2 + \frac{|\omega|}{\alpha}} y} + A_2 e^{-\sqrt{k_x^2 + \frac{|\omega|}{\alpha}} y} + A_3 e^{\sqrt{k_x^2 - \frac{|\omega|}{\alpha}} y} + A_4 e^{-\sqrt{k_x^2 - \frac{|\omega|}{\alpha}} y} + \frac{\widehat{p}}{K\left[k_x^4 - \left(\frac{\omega}{\alpha} \right)^2 \right]}$$

$$\widehat{\varphi}_x(k_x, y, \omega) := \frac{\widehat{w}(k_x, y, \omega)}{\partial y} \quad , \quad \widehat{\varphi}_y(k_x, y, \omega) = jk_x \widehat{w}(k_x, y, \omega) \tag{34}$$

$$\widehat{\varphi}_x(k_x, y, \omega) = \sqrt{k_x^2 + \frac{|\omega|}{\alpha}} \left(A_1 e^{\sqrt{k_x^2 + \frac{|\omega|}{\alpha}} y} - A_2 e^{-\sqrt{k_x^2 + \frac{|\omega|}{\alpha}} y} \right) + \sqrt{k_x^2 - \frac{|\omega|}{\alpha}} \left(A_3 e^{\sqrt{k_x^2 - \frac{|\omega|}{\alpha}} y} - A_4 e^{-\sqrt{k_x^2 - \frac{|\omega|}{\alpha}} y} \right)$$

$$\widehat{m}_{xx} := -K\left(-k_x^2 \widehat{w} + \nu \frac{\partial^2 \widehat{w}}{\partial y^2} \right) \ , \ \widehat{m}_{yy} := -K\left(\frac{\partial^2 \widehat{w}}{\partial y^2} - \nu k_x^2 \widehat{w} \right) \ , \ m_{xy} := -K(1-\nu) jk_x \frac{\partial \widehat{w}}{\partial y}$$

$$\widehat{q}_x := jk_x m_{xx} + \frac{\partial m_{xy}}{\partial y} \ , \ \widehat{q}_y := \frac{\partial m_{yy}}{\partial y} + jk_x m_{xy}$$

$$\widehat{m}_{yy} = -K \begin{pmatrix} \left(k_x^2(1-\nu) + \frac{|\omega|}{\alpha} \right) \left(A_1 e^{\sqrt{k_x^2 + \frac{|\omega|}{\alpha}} y} + A_2 e^{-\sqrt{k_x^2 + \frac{|\omega|}{\alpha}} y} \right) + \\ + \left(k_x^2(1-\nu) - \frac{|\omega|}{\alpha} \right) \left(A_3 e^{\sqrt{k_x^2 - \frac{|\omega|}{\alpha}} y} + A_4 e^{-\sqrt{k_x^2 - \frac{|\omega|}{\alpha}} y} \right) \end{pmatrix} + \frac{\nu k_x^2 \widehat{p}}{\left[k_x^4 - \left(\frac{\omega}{\alpha} \right)^2 \right]} \tag{35}$$

$$\widehat{q}_y = -K \frac{\omega}{\alpha} \begin{pmatrix} \sqrt{k_x^2 + \frac{|\omega|}{\alpha}} \left(A_1 e^{\sqrt{k_x^2 + \frac{|\omega|}{\alpha}} y} - A_2 e^{-\sqrt{k_x^2 + \frac{|\omega|}{\alpha}} y} \right) - \\ - \sqrt{k_x^2 - \frac{|\omega|}{\alpha}} \left(A_3 e^{\sqrt{k_x^2 - \frac{|\omega|}{\alpha}} y} - A_4 e^{-\sqrt{k_x^2 - \frac{|\omega|}{\alpha}} y} \right) \end{pmatrix}$$

$$y = \pm l/2$$

Only the moment m_{yy} and the lateral force q_y are needed in the simulation. The torsional moment m_{xy} has to be neglected with respect to the Kirchhoff model and the remaining moment m_{xx} and force q_x do not act on the interface between the elements. In the same manner, only the displacement w and the rotation φ_x are needed.

5.2 Plane Shell Element

The equilibrium of the stresses of the plane shell element are defined by

$$\begin{aligned}\frac{\partial}{\partial x}\sigma_x+\frac{\partial}{\partial y}\tau_{xy}-\rho d\frac{\partial^2}{\partial t^2}u=p_x\\\frac{\partial}{\partial y}\sigma_y+\frac{\partial}{\partial x}\tau_{xy}-\rho d\frac{\partial^2}{\partial t^2}v=p_y\end{aligned}\tag{36}$$

The mass density is again ρ, the Young's modulus E and the Poisson's ration ν. The displacement in x is named u and the displacement in y is v. The shear pressures are p_x and p_y. The stress strain relationship is given by

$$\begin{bmatrix}\sigma_x\\\sigma_y\\\tau_{xy}\end{bmatrix}=\frac{E}{1-\nu^2}\begin{bmatrix}1&\nu&0\\\nu&1&0\\0&0&(1-\nu)/2\end{bmatrix}\begin{bmatrix}\varepsilon_x\\\varepsilon_y\\\gamma_{xy}\end{bmatrix},\tag{37}$$

where

$$\begin{bmatrix}\varepsilon_x\\\varepsilon_y\\\gamma_{xy}\end{bmatrix}=\begin{bmatrix}\partial u/\partial x\\\partial v/\partial y\\\partial u/\partial y+\partial v/\partial x\end{bmatrix}.\tag{38}$$

Using Eq. 36, 37 and 38 equations that only depends on the displacements are derived:

$$\begin{aligned}&\frac{E}{1-\nu^2}\left[\left(\frac{\partial^2u}{\partial x^2}+\nu\frac{\partial^2v}{\partial x\partial y}\right)+\frac{1-\nu}{2}\left(\frac{\partial^2u}{\partial y^2}+\frac{\partial^2v}{\partial x\partial y}\right)\right]-\rho d\frac{\partial^2u}{\partial t^2}=-p_x\\&\frac{E}{1-\nu^2}\left[\left(\frac{\partial^2v}{\partial y^2}+\nu\frac{\partial^2u}{\partial x\partial y}\right)+\frac{1-\nu}{2}\left(\frac{\partial^2v}{\partial x^2}+\frac{\partial^2u}{\partial x\partial y}\right)\right]-\rho d\frac{\partial^2v}{\partial t^2}=-p_y\\&B=\frac{E}{1-\nu^2}\quad,\quad\beta=\sqrt{\frac{B}{\rho d}}=\sqrt{\frac{E}{\rho d(1-\nu^2)}}\\&\left[\left(\frac{\partial^2u}{\partial x^2}+\nu\frac{\partial^2v}{\partial x\partial y}\right)+\frac{1-\nu}{2}\left(\frac{\partial^2u}{\partial y^2}+\frac{\partial^2v}{\partial x\partial y}\right)\right]-\frac{1}{\beta^2}\frac{\partial^2u}{\partial t^2}=\frac{-p_x}{B}\\&\left[\left(\frac{\partial^2v}{\partial y^2}+\nu\frac{\partial^2u}{\partial x\partial y}\right)+\frac{1-\nu}{2}\left(\frac{\partial^2v}{\partial x^2}+\frac{\partial^2u}{\partial x\partial y}\right)\right]-\frac{1}{\beta^2}\frac{\partial^2v}{\partial t^2}=\frac{-p_y}{B}.\end{aligned}\tag{39}$$

The Bulk modulus B is used to shorten the equations. The following equation is derived applying Fourier integral transformation with respect to the longitudinal direction x and time t.

$$\begin{aligned}\left[\left(-k_x^2\hat{u}+\nu jk_x\frac{\partial}{\partial y}\hat{v}\right)+\frac{1-\nu}{2}\left(\frac{\partial^2}{\partial y^2}\hat{u}+jk_x\frac{\partial}{\partial y}\hat{v}\right)\right]+\left(\frac{\omega}{\beta}\right)^2\hat{u}=\frac{-\hat{p}_x}{B}\\\left[\left(\frac{\partial^2}{\partial y^2}\hat{v}+\nu jk_x\frac{\partial}{\partial y}\hat{u}\right)+\frac{1-\nu}{2}\left(-k_x^2\hat{v}+jk_x\frac{\partial}{\partial y}\hat{u}\right)\right]+\left(\frac{\omega}{\beta}\right)^2\hat{v}=\frac{-\hat{p}_y}{B}\end{aligned}\tag{40}$$

The particular solution is again derived assuming that the deflection does not depend on the coordinate y.

$$-k_x^2\hat{u}_0+\left(\frac{\omega}{\beta}\right)^2\hat{u}_0=\frac{-\hat{p}_x}{B} \quad , \quad -\frac{1-\nu}{2}k_x^2\hat{v}_0+\left(\frac{\omega}{\beta}\right)^2\hat{v}_0=\frac{-\hat{p}_y}{B}$$

$$\hat{u}_0=\frac{\hat{p}_x}{B\left(k_x^2-\left(\frac{\omega}{\beta}\right)^2\right)} \quad , \quad \hat{v}_0=\frac{\hat{p}_y}{B\left(\frac{1-\nu}{2}k_x^2-\left(\frac{\omega}{\beta}\right)^2\right)} \tag{41}$$

The following particular solutions are possible:

$$\begin{aligned}
jk_x\hat{u}&=\frac{jk_x\hat{p}_x}{B\left(k_x^2-\left(\frac{\omega}{\beta}\right)^2\right)}\\
\partial\hat{u}/\partial y&=0\\
jk_x\hat{v}&=\frac{jk_x\hat{p}_y}{B\left(\frac{1-\nu}{2}k_x^2-\left(\frac{\omega}{\beta}\right)^2\right)}\\
\partial\hat{v}/\partial y&=0\,.
\end{aligned} \tag{42}$$

With these relations the stresses can be derived:

$$\begin{bmatrix}\hat{\sigma}_x\\ \hat{\sigma}_y\\ \hat{\tau}_{xy}\end{bmatrix}=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & (1-\nu)/2\end{bmatrix}\begin{bmatrix}jk_x\hat{u}\\ 0\\ jk_x\hat{v}\end{bmatrix}=\frac{E}{1-\nu^2}\begin{bmatrix}\dfrac{jk_x\hat{p}_x}{B\left(k_x^2-\left(\frac{\omega}{\beta}\right)^2\right)}\\ \dfrac{\nu jk_x\hat{p}_x}{B\left(k_x^2-\left(\frac{\omega}{\beta}\right)^2\right)}\\ \dfrac{(1-\nu)jk_x\hat{p}_y}{2B\left(\frac{1-\nu}{2}k_x^2-\left(\frac{\omega}{\beta}\right)^2\right)}\end{bmatrix}. \tag{43}$$

The homogenous solution is derived from the following matrix equation

$$\begin{aligned}
&\left(-k_x^2\breve{u}-\nu k_xk_y\breve{v}\right)+\frac{1-\nu}{2}\left(-k_y^2\breve{u}-k_xk_y\breve{v}\right)+\left(\frac{\omega}{\beta}\right)^2\breve{u}=0\\
&\left(-k_y^2\breve{v}-\nu k_xk_y\breve{u}\right)+\frac{1-\nu}{2}\left(-k_x^2\breve{v}-k_xk_y\breve{u}\right)+\left(\frac{\omega}{\beta}\right)^2\breve{v}=0\\
&\begin{bmatrix}k_x^2+\frac{1-\nu}{2}k_y^2-\left(\frac{\omega}{\beta}\right)^2 & \frac{1+\nu}{2}k_xk_y\\ \frac{1+\nu}{2}k_xk_y & k_y^2+\frac{1-\nu}{2}k_x^2-\left(\frac{\omega}{\beta}\right)^2\end{bmatrix}\begin{bmatrix}\breve{u}\\ \breve{v}\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}.
\end{aligned} \tag{44}$$

The matrix becomes singular at

$$\frac{1-\nu}{2}\left(k_x^4+2k_x^2k_y^2+k_y^4\right)-\frac{3-\nu}{2}\left(k_x^2+k_y^2\right)\left(\frac{\omega}{\beta}\right)^2+\left(\frac{\omega}{\beta}\right)^4=0$$

$$a=\frac{1-\nu}{2}\quad,\quad b=(1-\nu)k_x^2-\frac{3-\nu}{2}\left(\frac{\omega}{\beta}\right)^2\quad,\quad c=\frac{1-\nu}{2}k_x^4-\frac{3-\nu}{2}k_x^2\left(\frac{\omega}{\beta}\right)^2+\left(\frac{\omega}{\beta}\right)^4 \tag{45}$$

$$k_{y,1,2}^2=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Overall, four points for the singularity exist and the following eigenvectors are derived

$$k_{y,1,2,3,4}=\pm\sqrt{k_{y1,2}^2}\Rightarrow\Psi_{1,2,3,4}$$
$$\begin{bmatrix}\hat{u}(y)\\\hat{v}(y)\end{bmatrix}=\sum_{i=1}^{4}C_i\Psi_i e^{jk_{y,i}y}\quad,\quad y=\pm\frac{l}{2} \tag{46}$$

Again, the generalized eigenvectors of the stresses are needed for completeness.

$$\begin{bmatrix}\hat{\sigma}_{x,i}\\\hat{\sigma}_{y,i}\\\hat{\tau}_{xy,i}\end{bmatrix}=B\begin{bmatrix}1&\nu&0\\\nu&1&0\\0&0&(1-\nu)/2\end{bmatrix}\begin{bmatrix}-jk_x\hat{u}_i\\-jk_{y,i}\hat{v}_i\\-jk_{y,i}\hat{u}_i-jk_x\hat{v}_i\end{bmatrix}$$

$$\Phi_i=B\begin{bmatrix}-jk_x&-\nu jk_{y,i}\\-\nu jk_x&-jk_{y,i}\\-(1-\nu)/2\,jk_{y,i}&-(1-\nu)/2\,jk_x\end{bmatrix}\Psi_i \tag{47}$$

$$\begin{bmatrix}\hat{\sigma}_x(y)\\\hat{\sigma}_y(y)\\\hat{\tau}_{xy}(y)\end{bmatrix}=\sum_{i=1}^{4}C_i\Phi_i e^{jk_{y,i}y}\quad,\quad y=\pm\frac{l}{2}$$

6. COUPLING OF BEM WITH THE PLANE SHELL ELEMENT IN 2.5D

As already mentioned, the stresses at the boundary of the soil are coupled with the pressure and shear loads on the shell element. Constant pressures are assumed for this case. Therefore, constant shape functions are used in the BEM part.

The stiffness matrix for every element k of the FEM part $\mathbf{K}_{e,k}$ is derived by producing two matrices for the homogenous part and two matrices for the particular part. One matrix $\mathbf{M}_{F,w,k}$ contains the dependency of the deformations on the internal degrees of freedom (A_i and C_i) in Eq. 34 and Eq. 46 at the endpoints of the element $l/2$ and $-l/2$, this matrix is inverted and multiplied with the matrix that contains the dependency of the forces on the internal degrees of freedom $\mathbf{M}_{F,f,k}$ in Eq. 35 and Eq. 47. In this way, a dynamic stiffness matrix of the elements is desired. The matrix $\mathbf{M}_{F,wp,k}$ with dependency of the displacements on the pressure loads p and the matrix with the dependency of the forces with the pressure loads $\mathbf{M}_{F,fp,k}$ are build in the same way using Eq. 34, Eq. 35, Eq.41 and Eq.43.

$$\begin{aligned}&\left[u\left(-\tfrac{l}{2}\right)\ \ v\left(-\tfrac{l}{2}\right)\ \ w\left(-\tfrac{l}{2}\right)\ \ \varphi_x\left(-\tfrac{l}{2}\right)\ \ u\left(+\tfrac{l}{2}\right)\ \ v\left(+\tfrac{l}{2}\right)\ \ w\left(+\tfrac{l}{2}\right)\ \ \varphi_x\left(+\tfrac{l}{2}\right)\right]^T=\\&=\mathbf{M}_{F,w,k}\left[C_1\ \ C_2\ \ C_3\ \ C_4\ \ A_1\ \ A_2\ \ A_3\ \ A_4\right]^T\\&\left[\sigma_y\left(-\tfrac{l}{2}\right)\ \ \tau_{xy}\left(-\tfrac{l}{2}\right)\ \ q\left(-\tfrac{l}{2}\right)\ \ m_x\left(-\tfrac{l}{2}\right)\ \ \sigma_y\left(+\tfrac{l}{2}\right)\ \ \tau_{xy}\left(+\tfrac{l}{2}\right)\ \ q\left(+\tfrac{l}{2}\right)\ \ m_x\left(+\tfrac{l}{2}\right)\right]^T=\\&=\mathbf{M}_{F,f,k}\left[C_1\ \ C_2\ \ C_3\ \ C_4\ \ A_1\ \ A_2\ \ A_3\ \ A_4\right]^T\end{aligned} \tag{48}$$

The first matrix is multiplied with the stiffness matrix and added with the second matrix to build up a load matrix of the elements. The elemental stiffness matrix and the load matrix are rotated with $\mathbf{R}_k$ into global coordinates ($\mathbf{u}_g$) and added to a global stiffness matrix of the FEM part. Again the load matrix is multiplied by a vector of the external pressure load on the FEM elements to build up the global load vector.

The coupling of the displacements is done at the nodes. Here, linear shape functions are used in the BEM part but analytic solutions are used for the plane shell elements. Of course, the stresses and moments are coupled from one shell element to the next at the nodes independent from the BEM part.

For the coupling the BEM part is solved for a constant unit valued pressure on every element. The solution is a matrix $\mathbf{M}_B$ for the displacements at the nodes depending on the pressure loads. This matrix is inverted to receive a dynamic stiffness matrix for the BEM part. The matrix is multiplied with the pressure displacement relationship and the pressure force relationship of the FEM part. The pressure displacement relationship is converted into a pressure force relationship using again the stiffness matrix of the FEM part using the rotation matrix with $\mathbf{R}_B$. The result is a global dynamic stiffness matrix of the BEM part.

The dynamic stiffness matrix is added to the global dynamic stiffness matrix of the FEM part. The global matrix of both parts is solved and the unknown displacements at the interface are determined.

$$\begin{gathered} \mathbf{K}_{e,k} = \mathbf{M}_{F,f,k}\mathbf{M}_{F,w,k}^{-1} \quad , \quad \mathbf{M}_F = \sum_k \left(\mathbf{R}_k^T \left[\mathbf{M}_{F,fp,k} - \mathbf{K}_{e,k}\mathbf{M}_{F,wp,k}\right]\right) \\ \left\{ \sum_k \left(\mathbf{R}_k^T \mathbf{K}_{\acute{e},k} \mathbf{R}_k\right) + \mathbf{M}_F \mathbf{M}_B^{-1} \mathbf{R}_B \right\} \mathbf{u}_g = -\mathbf{M}_F \mathbf{p}_E \end{gathered} \tag{48}$$

The last step is the extrapolation of the displacements at the surface of the layered half space. From the FEM matrix, the pressures at the elements are derived using the known nodal displacements. To derive these displacements at the surface a transfer matrix of the desired displacements depending on the unit pressures at the interface is derived together with the other matrix of the BEM part. This matrix is multiplied with the known pressure loads from the step before to receive the desired vibrations at the surface or anywhere else. The remaining step is a numerical inverse Fourier integral transformation with respect to the wave number k_x.

7. CONCLUSIONS AND OUTLOOK

A method is described that allows using a horizontally stratified medium as the domain for a boundary element method. The stratification allows for reflections and transmission of interface waves. These effects often dominate the response at the surface for position in larger distance.

The coupling of BEM with analytic solutions for the shell elements is rather inconvenient, but allows for short wavelength that really occur in the shell elements but do not radiate into the surrounding. The assumption of constant pressures for the coupling with the BEM part filters the parts with short wavelengths.

REFERENCES

1. Abd-Alla, A. M., “Propagation of Rayleigh waves in an elastic half-space othotropic material”. Applied Mathematics and Computation, March 1999, Volume 99 Issue 1
2. Abd-Alla, A. M., Ahmed, S. M., Propagation of Love waves in a non-homogeneous orthotropic elastic layer under initial stress overlying semi-infinite medium. Applied Mathematics and Computation, December 1999, Volume 106 Issue 2-3
3. Andersen, L., Jones, C.J.C. Vibration from a railway tunnel predicted by coupled finite element and boundary element analysis in two and three dimensions, Structural Dynamics, Eurodyn 2002, 1131-1136
4. Antes, H., Borejko, P., Ziegler, F.: Seismic waves in an impacted half-space: Time domain BEM versus the method of generalized ray, Boundary Elements XIII, 1991, 457-470
5. Borejko, P., Ziegler, F. Seismic Waves in layered soil: The generalized ray theory, Structural Dynamics - Recent Advances, 1991, 52-90
6. Degrande, G., Wave propagation in the soil: theoretical background and application to traffic induced vibrations. Eurodyn 2002, Swets & Zeitlinger, Lisse 2002, 27-40
7. Dinkel, J., Ein semi-analytisches Modell zur dynamischen Berechnung des gekoppelten Systems Fahrzeug-Fahrweg-Untergrund für das Oberbausystem Feste Fahrbahn. Dissertation, Technische Universität München 2000
8. Duddeck, F. M. E. Fourier-BEM; or what to do if no fundamental solution is available? Meccanica, 36,

2001, 437-448
9. Gaul, L., Kögl, M. Wagner, M. Boundary Element Methods for Engineers and Scientists, An Introductory Course with Advanced Topics, Springer, Berlin Heidelberg, 2003
10. Hartmann, F.: Methode der Randelemente, Boundary Elements in der Mechanik auf dem PC, Springer, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1987
11. Hosten, B. Reflection and Transmission of Acoustic Plane Waves on an Immersed Orthotropic and Viscoelastic Solid Layer. Journal of the Acoustic Section of America 89, 1991, 2745-2752
12. Hung, H.-H., Yang, Y.-B. A review of researches on ground-borne vibrations with emphasis on those induced by trains, Proc. Natl. Sci. Counc. ROC(A), 25(1), 2001, 1-16
13. Lieb, M. Adaptive numerische Fouriertransformation in der Bodendynamik unter Verwendung einer Waveletzerlegung. Dissertation, Technische Universität München 1997
14. Liu, K., Li, X., Tan, Y., Tanimura, S. Stress Wave Propagation in a Layered Orthotropic Medium. Proceedings of the 3rd International Symposium on IMPACT ENGINEERING'98, Singapore, 1998
15. Müller, G. Ein Verfahren zur Erfassung der Fundament-Boden-Wechselwirkung unter der Einwirkung periodischer Lasten. Dissertation, Technische Universität München 1989
16. Nayfeh, A.H. The general problem of elastic wave propagation in multilayered anisotropic media, J. Acoust. Soc. Am., 89(4), 1991, 1521-1531
17. Rieckh, G., Kreuzer, W., Waubke, H., Balazs, P. A 2.5D-Fourier-BEM-model for vibrations in a tunnel running through layered anisotropic soil, Engineering Analysis with Boundary Elements 36, 2012, 960-967.
18. Ruge, P., Trinks, C., Consistent time-domain models for unbounded space-domains. Fifth World Congress on Computational Mechanics, Vienna, 2002, http://wccm.tuwien.ac.at
19. Savidis, S.A., Bode, C., Dynamic soil track interaction. Eurodyn 2002, Swets & Zeitlinger, Lisse 2002
20. Tewary V.K., Mahapatra, M., Fortunko, C.M. Green's function for anisotropic half-space solids in frequency space and calculation of mechanical admittance, J. Acoust. Soc. Am., 100(5), 1996, 2960-2963
21. Trinks, C., Ruge, P. "Description of wave propagation by fading memory", Eurodyn 2002, Swets & Zeitlinger, Lisse 2002
22. Waubke, H. Dynamische Berechnungen für den Halbraum mit streuenden Parametern mittels orthogonaler Polynome. Dissertation, Technische Universität München 1996
23. Waubke, H. Numerical solutions for waves in linear anisotropic media, 18th Int. Congress on Acoustics, 2004
24. Waubke, H., Vibrations in layered orthotropic soils, 1st Congress of the Alps Adria Acoustic Association, Graz, 2003.
25. Waubke, H. Transform Methods for Horizontally Layered Isotropic and Anisotropic Media with Obstacles, DAGA, 2004.
26. Waubke, H. Waves in a layered orthotropic medium with and without random properties. 10th Int. Congress on sound and vibration, Stockholm, 2003
27. Waubke, H., Kreuzer, W. Dynamic Load in a Layered Soil With Anisotropic Material. AIA-DAGA 2013.
28. Waubke, H., Kreuzer, W. Kopplung von finiten Elementen mit Randelementen im Orts-Wellenzahlraum zur Simulation von Tunnelstrukturen, DAGA 2014.
29. Waubke, H., Kreuzer, W., Vibrations from railways in tunnels and trenches with elastic anisotropic layered mediums, 3rd Congress of the Alps Adria Acoustic Association, Graz, 2007.
30. Waubke H., Kreuzer, W. Lastansätze innerhalb einer Schicht eines geschichteten anistropen Mediums, DAGA 2008.
31. Waubke, H., Kreuzer, W., Balazs, P. Ein 3D-Modell zur Berechnung der Ausbreitung von Erschütterungen in Bodenschichten, DAGA 2006.
32. Waubke, H., Kreuzer, W., Rieckh, G. Teilanalytische Lösung der Integrale der Randelemente Methode mit numerisch bestimmter Greenscher Funktion für geschichtete anisotrope Medien, DAGA 2011
33. Wolf, J.P. Dynamic Soil-Structure Interaction. Eaglewood Cliffs N.J., Prentice-Hall 1985
34. Zirwas, G. Ein hybrides Verfahren zur Behandlung der Bauwerk-Boden-Wechselwirkung mit analytischen Integraltransformationen und numerischen Ansätzen, Dissertation, Technische Universität München 1995