IMPLEMENTATION OF A NUMERICAL METHOD FOR THE BEST FITTING OF THE BENDING STIFFNESS CURVE TO A SET OF EXPERIMENTAL POINTS

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ABSTRACT

Industries working in the automotive and building fields are interested in the prediction and the determination of the sound reduction index given by lightweight partitions. The investments in this research field are aiming to find experimental methods capable to assess the sound insulation performances of small specimens instead of full scale partitions. Using software packages based on FEM-BEM methods, some variables deriving from the manufacturing stage and influencing the sound transmission index are not taken into account. In the last decades a few methods based on the determination of the dynamic properties of sandwich beams have been developed and validated. The actual bending stiffness of the beam specimen can be determined on the basis of the least square method applied to a set of frequency dependent bending stiffness points determined for each natural frequency of a beam tested in free-free conditions. For this reason, it is of interest to examine the method used to perform a successful curve fitting. The aim of the present paper is to investigate possible alternative approaches to the solution of the computational problem.

Keywords: Bending stiffness, Sound Reduction Index, Least square method
I-INCE Classification of Subjects Numbers: 23.9; 43.2

1. INTRODUCTION

The dynamic properties of a composite beam or plate consisting of a foam core and two external plates depend on the geometry of the assembly and on the physical properties of core and external leaves. In some cases, such as the experimental one described in the following section, the method of bonding the laminates to the core can influence the thickness of the laminates and then the dynamic properties of the sandwich construction.

As Described by Nilsson (1), some of the basic parameters of a sandwich structure can be determined by means of simple tests carried out on a beam element of the structure. The beam is suspended by strings to simulate free-free boundary conditions. By exciting the beam using an impedance hammer the natural frequencies can be determined. Based on these measurements of natural frequencies, the mass per unit area of the specimen and dimensions of the beam, the frequency dependent bending stiffness of the beam can be computed.

The vibrations of a sandwich structure put in motion by an external sound field have been discussed in several papers. Some references are (2) and (3). In deriving the equations governing the motion of the structure symmetry is assumed (Figure 1). The laminates have a Young's modulus E_2, bending stiffness D_2, density ρ_l and thickness h. The core has a shear stiffness G_e, a Young´s modulus E_1, a density ρ_c and a thickness H. In general $E_2 >> E_1$. The core is assumed to be stiff enough to ensure the laminates to move in phase. The bending stiffness per unit width of the beam is

$$D_1 = E_1 H^3 / 12 + E_2 (H^2 h / 2 + H h^2 + 2h^3 / 3) \tag{1}$$

The bending stiffness of one laminate is

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$$D_2 = E_2 h^3 / 12 \tag{2}$$

The mass moment of inertia per unit width is defined as

$$I_\rho = \rho_c H^3 / 12 + \rho_l (H^2 h / 2 + H h^2 + 2h^3 / 3) \tag{3}$$

while the mass per unit area is

$$\mu = 2h\rho_l + H\rho_c \tag{4}$$

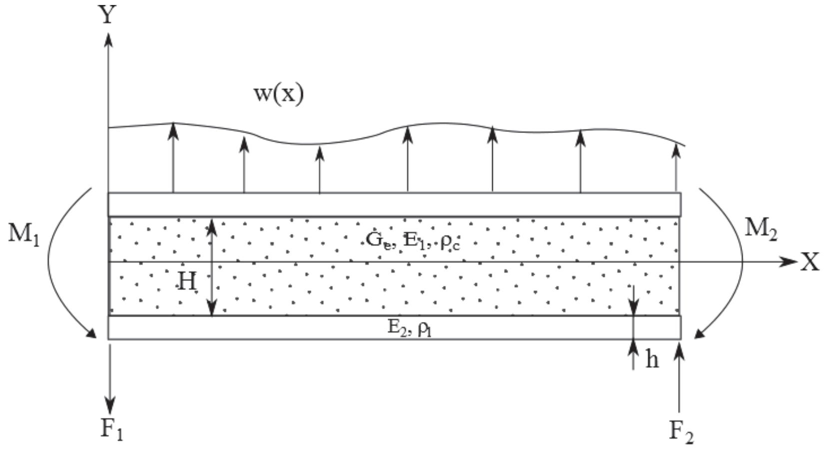

Figure 1 – Sketch of the cross section of a beam, forces and moments.

Hamilton's principle can be applied for deriving the equations governing the bending of the beam. The total potential energy is due to pure bending of the entire beam, bending of the laminates and shear stresses inside the core. The kinetic energy of the sandwich beam consists of, the kinetic energy due to motion of the beam and the kinetic energy due to rotation of the beam. According to (2), the free lateral displacement w of the beam is given by:

$$-2D_1 D_2 \frac{\partial^6 w}{\partial x^6} + 2D_2 I_\rho \frac{\partial^6 w}{\partial x^4 \partial t^2} - \left(D_1 \mu + 2D_2 \mu + I_\rho G_e H\right) \frac{\partial^4 w}{\partial x^2 \partial t^2} + G_e H \left(D_1 \frac{\partial^4 w}{\partial x^4} + \mu \frac{\partial^2 w}{\partial t^2} \right) + + I_\rho \mu \frac{\partial^4 w}{\partial t^4} = 0 \tag{5}$$

The described method is applicable to beams. In case of more complex structures having finite dimensions, the method can be adapted by using the space and frequency average of the point mobility, measured on a set of positions randomly distributed over the surface of the panel, as discussed in (4).

2. WAVENUMBERS AND APPARENT BENDING STIFFNESS

If the expression $w = \exp[i(\omega t - k_x x)]$ is a solution to Eq. (5), it is possible to derive the relevant wavenumbers. The solutions are six. Two of them, $k_x = \pm\kappa_1$, describe propagating waves, while the other solutions characterise evanescent waves. The apparent bending stiffness D_x can be defined as the bending stiffness of a simple homogeneous beam having, at a certain frequency, the same dynamic properties of the considered sandwich structure. The wavenumber for the first propagating wave can be written as a function of the apparent bending stiffness as $(k_x)^4 = \omega^2\mu/D_x$. By inserting the definition of k_x in Eq. (5), an equation in D_x is obtained. The resulting expression can generally be simplified if the bending stiffness of the entire beam is far greater than the bending stiffness of one laminate and if the potential energy of the core is much greater than the kinetic energy of the beam. For the type of sandwich structure used for the experimental tests, these assumptions hold within the frequency range of interest. The apparent bending stiffness D_x is obtained as the solution to the following equation:

$$\left(\frac{G_e H}{\mu^{1/2}\omega}\right)\left[\frac{D_x^{3/2}}{D_1} - D_x^{1/2}\right] + D_x - 2D_2 = 0 \qquad (6)$$

In the low frequency range the first part of the equation rules and the bending stiffness is equal to the bending stiffness of the entire beam. In the high frequency range, the apparent bending stiffness is equal to $2D_2$. As a consequence, the bending stiffness for the entire beam is twice the bending stiffness of one single laminate.

For a beam suspended in free-free boundary conditions, the bending stiffness can be determined by applying a simple measurement technique. For a beam having a length L and a mass per unit area μ, the apparent bending stiffness D_{xn} for a normal mode n placed at a particular Eigen-frequency f_n, is

$$D_{xn} = 4\pi^2 f_n^2 \mu L^4 / \alpha_n^4 \qquad (7)$$

The values of α_n are given in Table 1 as a function of the n-th mode.

Table 1 – Values assumed by α_n as a function of the n-th mode.

n	1	2	3	4	≥5
α_n	4.73	7.85	11.0	14.14	$n\pi + \pi/2$

As given by Eq. (6) the bending stiffness of a sandwich beam is frequency dependent. The same Eq. (6) can be expressed as

$$\frac{A}{f} D_x^{3/2} - \frac{B}{f} D_x^{1/2} + D_x - C = 0 \qquad (8)$$

where

$$A = \frac{G_e H}{\mu^{1/2} 2\pi D_1}; \quad B = \frac{G_e H}{\mu^{1/2} 2\pi}; \quad C = 2D_2 \qquad (9)$$

For a beam made by a non-metallic material Young's modulus can be slightly frequency dependent. Nevertheless below 5000 Hz the parameters D_1, D_2 and G_e can be assumed to be almost constant. The parameters *A, B* and *C* can be determined by means of the least square method, as described in (2).

In the present paper, the numerical problem has been formulated by using Eq. (8) to extract the expression of *f*:

$$f = \frac{AD^{3/2} - BD^{1/2}}{C - D} \qquad (10)$$

By introducing the two sets of experimental data $\{f_i\}$ and $\{D_i\}$ with $i = 1,\dots, N$ in Eq. (10), one obtains:

$$f_i = \frac{AD_i^{3/2} - BD_i^{1/2}}{C - D_i} \qquad (11)$$

It is now possible to define the distance between the experimental natural frequencies and the frequencies described by Eq. (11), r_i:

$$r_i(A,B,C) = \frac{AD_i^{3/2} - BD_i^{1/2}}{C - D_i} - f_i \quad \text{with } i = 1,\dots, N \qquad (12)$$

and then to minimise r_i:

$$\min_{A,B,C} \left\| r_i(A,B,C) \right\|_{L_D^2}^2 = \min_{A,B,C} \sum_i \left(r_i(A,B,C) \right)^2 \qquad (13)$$

Once vector **x** is defined as:

$$\mathbf{x} = \begin{bmatrix} x(1) \\ x(2) \\ x(3) \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \end{bmatrix} \tag{14}$$

the quantity to be minimised can be expressed as the sum of the distances which are a function of A, B and C, and therefore of **x**:

$$\sum_i (r_i(\mathbf{x}))^2 = \left[r_1^2(\mathbf{x}) + r_2^2(\mathbf{x}) + \ldots + r_N^2(\mathbf{x})\right] = \|\mathbf{R}(\mathbf{x})\|^2 \tag{15}$$

where

$$\mathbf{R}(\mathbf{x}) = \begin{bmatrix} r_1(\mathbf{x}) \\ r_2(\mathbf{x}) \\ \vdots \\ r_N(\mathbf{x}) \end{bmatrix} \tag{16}$$

Function to be minimised,

$$\tilde{g}(\mathbf{x}) = \|\mathbf{R}(\mathbf{x})\|^2 = \sum_1^N \left(\frac{A \cdot D_i^{3/2} - B \cdot D_i^{1/2}}{C - D_i} - f_i \right)^2 \tag{17}$$

is concave up, so there is only one minimum point corresponding to zero gradient.
The Newton formula for non-linear systems can be described by the following set of equations:

$$\begin{cases} \mathbf{x_0} = [A_0 \; B_0 \; C_0] \\ \mathbf{x^{n+1}} = \mathbf{x_0} + \delta \mathbf{x^n} \end{cases} \tag{18}$$

where $\mathbf{x_0}$ is initial point,

$$\delta \mathbf{x^n} = -\mathbf{H}^{-1}(\mathbf{x}) \nabla \tilde{g}(\mathbf{x}) \tag{19}$$

$$\mathbf{H}(\mathbf{x}) = \mathbf{J}^T(\mathbf{x})\mathbf{J}(\mathbf{x}) + \mathbf{S}(\mathbf{x}) \tag{20}$$

$$\nabla \tilde{g}(\mathbf{x}) = \mathbf{J}^T(\mathbf{x})\mathbf{R}(\mathbf{x}) \tag{21}$$

However,

$$\nabla \tilde{g}(\mathbf{x}) = 0 \Rightarrow \begin{cases} \dfrac{d\tilde{g}(\mathbf{x})}{dx_1} = 0 \\ \dfrac{d\tilde{g}(\mathbf{x})}{dx_2} = 0 \\ \dfrac{d\tilde{g}(\mathbf{x})}{dx_3} = 0 \end{cases} \tag{22}$$

Since we are interested in finding a solution to equation $\mathbf{J}^T(\mathbf{x})\mathbf{R}(\mathbf{x})=0$, the system can be written as:

$$\begin{cases} \sum_{i}^{N} 2\left(\frac{AD_i^{3/2} - BD_i^{1/2}}{C - D_i} - f_i\right)\left(\frac{D_i^{3/2}}{C - D_i}\right) = 0 \\ \sum_{i}^{N} 2\left(\frac{AD_i^{3/2} - BD_i^{1/2}}{C - D_i} - f_i\right)\left(-\frac{D_i^{1/2}}{C - D_i}\right) = 0 \\ \sum_{i}^{N} 2\left(\frac{AD_i^{3/2} - BD_i^{1/2}}{C - D_i} - f_i\right)\left(-\frac{AD_i^{3/2} - BD_i^{1/2}}{C - D_i}\right) = 0 \end{cases} \quad (23)$$

The function $\mathbf{J}(\mathbf{x})$ can be defined for this problem as:

$$\mathbf{J}(\mathbf{x}) = \begin{bmatrix} \frac{dr_1}{dx_1} & \frac{dr_1}{dx_2} & \frac{dr_1}{dx_3} \\ \vdots & \vdots & \vdots \\ \frac{dr_N}{dx_1} & \frac{dr_N}{dx_2} & \frac{dr_N}{dx_3} \end{bmatrix} \quad (24)$$

whose elements can be computed as

$$\begin{cases} \frac{dr_i}{dx_1} = \left(\frac{D_i^{3/2}}{C - D_i}\right) \\ \frac{dr_i}{dx_2} = \left(-\frac{D_i^{1/2}}{C - D_i}\right) \\ \frac{dr_i}{dx_3} = \left(-\frac{AD_i^{3/2} - BD_i^{1/2}}{C - D_i}\right) \end{cases} \quad (25)$$

Considering a new function $\mathbf{f}$ defined as:

$$\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} \frac{d\tilde{g}(\mathbf{x})}{dx_1} \\ \frac{d\tilde{g}(\mathbf{x})}{dx_2} \\ \frac{d\tilde{g}(\mathbf{x})}{dx_3} \end{bmatrix} \quad (26)$$

the product $\mathbf{J}^T\mathbf{J}$ can be expressed as

$$\mathbf{J}^T\mathbf{J} = \begin{bmatrix} \frac{df_1}{dx_1} & \frac{df_1}{dx_2} & \frac{df_1}{dx_3} \\ \frac{df_2}{dx_1} & \frac{df_2}{dx_2} & \frac{df_2}{dx_3} \\ \frac{df_3}{dx_1} & \frac{df_3}{dx_2} & \frac{df_3}{dx_3} \end{bmatrix} = \begin{bmatrix} \frac{d^2\tilde{g}}{dx_1^2} & \frac{d^2\tilde{g}}{dx_1 dx_2} & \frac{d^2\tilde{g}}{dx_1 dx_3} \\ \frac{d^2\tilde{g}}{dx_2 dx_1} & \frac{d^2\tilde{g}}{dx_2^2} & \frac{d^2\tilde{g}}{dx_2 dx_3} \\ \frac{d^2\tilde{g}}{dx_3 dx_1} & \frac{d^2\tilde{g}}{dx_3 dx_2} & \frac{d^2\tilde{g}}{dx_3^2} \end{bmatrix} \quad (27)$$

Using the Gauss-Newton formula, rather than the Newton's method, the quantity $\mathbf{S}(\mathbf{x})$ in Eq. (20) is not needed and the non-linear least square problem can be solved easily.

In the following section the application of the method to a real case is shown.

3. APPLICATION OF THE CURVE FITTING METHOD TO A REAL CASE

3.1 Apparent bending stiffness of a beam

The method described in the previous section has been applied to a symmetric beam made up of a stiff 18 mm foam core and two fibre glass laminates, each one having a thickness of 1 mm. The beam is 2 m long and 50 mm high. The mass per unit area of the beam is 9.27 kg/m^2. In order to determine its apparent bending stiffness, the beam was suspended by strings to simulate free-free boundary conditions. The test procedure is described in (1). After this test, the first 31 modes and natural frequencies could be clearly identified (Figure 2). The repeatability of the measurements proved to be satisfactory. Measurements of the velocity on both laminates were recorded to verify that the laminates were moving in phase in the frequency range of interest, and the loss factor was determined accordingly, by using the half bandwidth method.

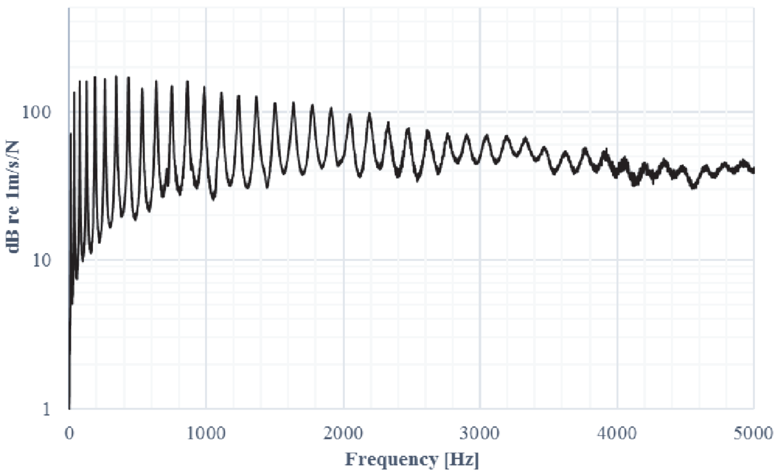

Figure 2 – Example of measured transfer function.

For each natural frequency f_n the corresponding apparent bending stiffness D_{xn} was calculated from Eq. (7). The experimental points have therefore been fitted by using the non-linear least square algorithm applied to Eq. (12). The result of the curve fitting is shown in Figure 3.

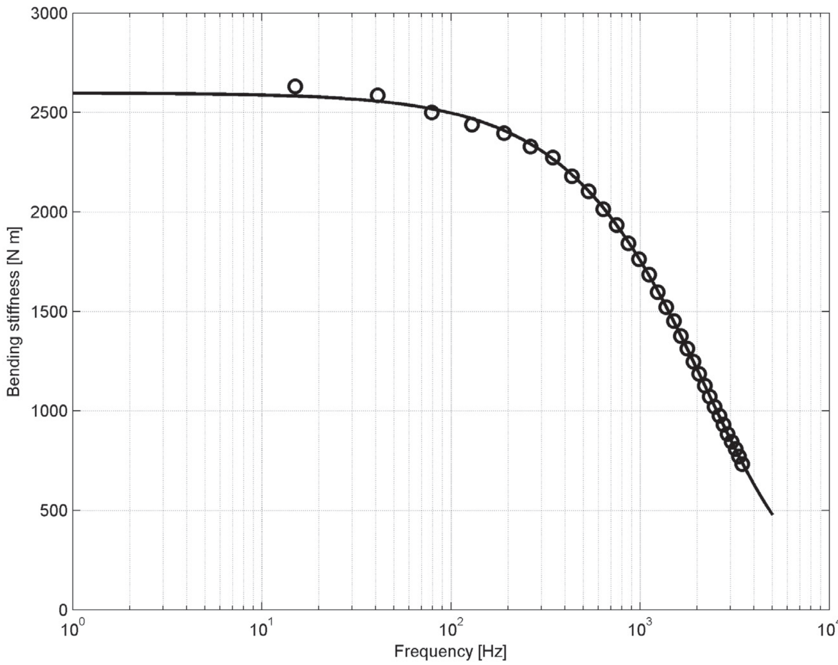

Figure 3 – Measured bending stiffness (circles) and predicted apparent bending stiffness of the sandwich beam (solid line).

3.2 Determination of the sound reduction index

The determination of the sound reduction index for single leaf panels is discussed for example in (5), and the expressions obtained for such structures can also be used for sandwich plates.

The sound reduction index R in decibels for a plate is, according to (5), $R = -10\log \tau_d$, where τ is the sound transmission coefficient for diffuse incidence. This coefficient is defined as

$$\tau_d = 2\int_0^{\pi/2} \tau(\varphi)\cos\varphi \sin\varphi d\varphi \tag{28}$$

The transmission coefficient $\tau(\varphi)$ at the angle of incidence φ is given by

$$\tau(\varphi) = \left\{ \left[1 + \frac{\mu\omega}{2\rho c}\cos\varphi \cdot \left(\frac{f}{f_c}\right)^2 (\sin\varphi)^4 \eta \right]^2 + \left[\frac{\mu\omega}{2\rho c} \cdot \cos\varphi \left\{ \left(\frac{f}{f_c}\right)^2 \cdot (\sin\varphi)^4 - 1 \right\} \right]^2 \right\}^{-1} \tag{29}$$

The parameters in Eq. (29) are: μ total mass per unit area of plate, f frequency, ω angular frequency, φ angle of incidence of acoustic wave, ρc wave impedance and η loss factor of structure.

f_c is the coincidence frequency, for which the trace matching between flexural waves on the plate and waves in the surrounding medium can occur. The frequency f_c for which $k_{plate} = k_{air}$ is given by

$$f_c = (c/2\pi)k_{plate}^2 / k_{air} = (c^2/2\pi)\sqrt{\mu/D} \tag{30}$$

where c is the speed of sound in air and, as a first approximation, k_{plate} can be set to equal κ_1 or the wavenumber for the first propagating mode of flexural waves in the plate. While for a thin single-leaf panel f_c is a constant, for a sandwich plate f_c is a function of frequency, since $D = D_x$ is a function of frequency and obtained from Eq. (8).

The sound reduction index for an infinite sandwich plate having the same structure as the composite beam used for the tests was predicted according to Eq. (28), using the bending stiffness curve described in the previous section. The results are shown in Figure 4.

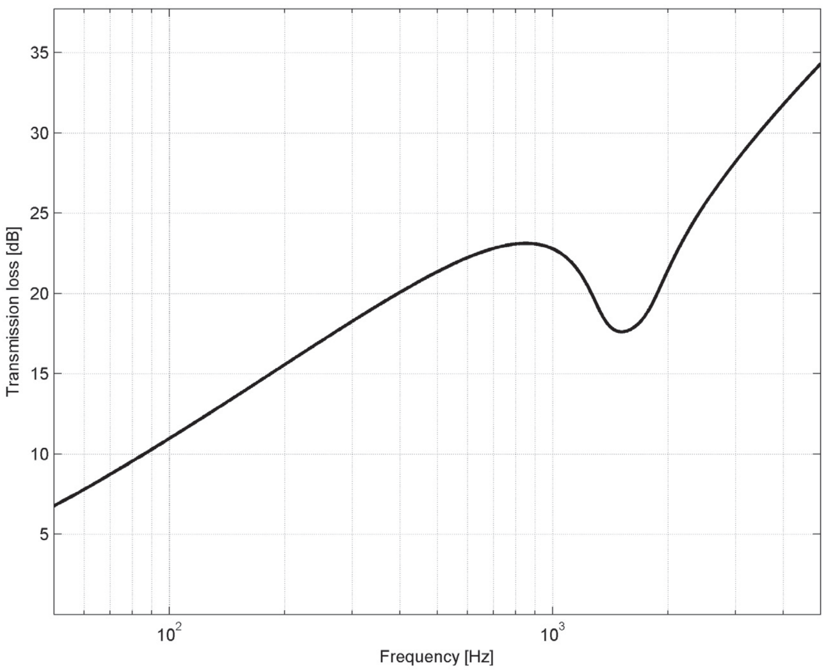

Figure 4 – Predicted sound reduction index for the sandwich plate.

4. CONCLUSIONS

The present paper investigates a least square approach for the determination of the frequency dependent bending stiffness curve of a beam, starting from a set of experimental values obtained from testing a beam specimen in free-free conditions.

The efficiency of the curve fitting is of particular importance, since the computed apparent bending stiffness is the main input for a mathematical model estimating the sound reduction index of the structure.

The application of the algorithm to a real specimen of sandwich beam showed very good curve fitting quality throughout the entire frequency range of interest.

In the future, measurements in sound transmission room will be performed on a panel having the same structure as the tested beam in order to compare the results of the prediction to experimental data.

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