Convergence of modes in exterior acoustics problems with infinite elements

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ABSTRACT
The radiated sound power is seldom computed by modal decomposition, since modal quantities are uncommon in exterior acoustics problems. The Finite Element Method and the Infinite Element Method (FEM and IFEM) are applied in order to discretize an unbounded fluid-filled domain and to obtain system matrices that are independent of frequency. From these system matrices of mass, damping and stiffness, frequency-independent normal modes are computed as right eigenvectors of a state-space eigenvalue problem. As the polynomial order of radial interpolation in the domain of the infinite elements increases, the normal mode eigenvalues converge and lead to reliable results for the radiated sound power in exemplary load cases. However, the additional degrees of freedom may also yield mathematical artifacts or spurious modes, which might falsify the calculated sound power in the case of modal superposition. By application of the Modal Assurance Criterion (MAC), significant and converged modes are identified and their contribution to the total radiated sound power is investigated.

Keywords: Exterior acoustics, Infinite Element Method (IFEM), normal modes

I-INCE Classification of Subjects Number(s): 75.6

1. INTRODUCTION

The acoustic exterior problem describes the unbounded sound pressure field filled with fluid around an inner obstacle. The Finite Element Method (FEM) is a common approach to discretizing and computing fluid-filled interior acoustic problems, and is described in detail in the literature [1]. However, if it is desirable for the outer boundaries of the domain to be non-reflective and the fluid to be of infinite size, FEM is no longer an appropriate approach. The Boundary Element Method (BEM) supports the calculation of the sound pressure on the structure’s surface as well as the radiated sound power for arbitrary structural velocities, considering non-reflective boundaries at infinity [2]. Other methods such as Perfectly Matched Layers (PML) [3] and the Infinite Element Method [4–6] also consider the non-reflective Sommerfeld radiation condition at infinity, but allow calculations in the fluid, not only on the surface of the obstacle. The conjugated Astley-Leis infinite elements provide the frequency-independent system matrices of stiffness, damping and mass similar to the FEM. For the radial interpolation in the domain of the infinite elements, polynomials such as Lagrange polynomials, Legendre polynomials (cf. Shirron and Babuška [7]) or Jacobi polynomials (cf. Dreyer and von Estorff [8, 9]) may be chosen. The latter provide the best-conditioned global system matrix and are also used by the authors of this paper.

The concept of normal modes was extensively presented by Marburg et al. [10–12] and allows the modal decomposition of an acoustic exterior domain. Therefore, the discrete linear system of equations containing the frequency-independent matrices of stiffness, damping and mass, are written in a state-space formulation. Solving the eigenvalue problem thus gives eigenvectors and eigenvalues which do not depend on the frequency. Moheit and Marburg [13] observed that the choice of the radial interpolation polynomials has hardly any effect on the eigenvalues of the normal modes. The eigenvalues are obtained from the diagonal of the matrix product of modal matrices and the frequency-independent state-space matrices, on which the normal modes are based. Because of this, orthogonality of the normal modes is desirable. Fuß et al. [14] studied search algorithms for normal mode eigenvalues with the example of a recorder. According to Marburg [11], the radiated sound power for arbitrary load cases can be calculated by modal superposition.

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The aim of this paper is the investigation of the influence of the orthogonality of the normal modes on the superimposed, radiated sound power. Here, the differences between interpolation polynomials of even and odd order are considered. These respectively lead to different convergences as the number of radial interpolation points is increased [13]. The studies in this work refer to the two-dimensional model of the cross section of a recorder.

2. THEORY

2.1 The acoustic boundary value problem

The acoustic exterior problem describes an unbounded fluid-filled domain of infinite extent around an inner obstacle. The spatial, continuous sound pressure field in the fluid can be described by the Helmholtz equation that is defined as

$$\nabla^2 p(x) + k^2 p(x) = 0, \quad x \in \Omega \subset \mathbb{R}^2,$$  

(1)

where $k = \omega/c_f$ is the wave number and $c_f$ is the speed of sound in the fluid. Neumann boundary conditions are assumed at the surface of the structure with a non-dissipative fluid-structure coupling. That means the structural particle velocity $v_s(x)$ is equal to the particle velocity in the fluid $v_f(x)$ on the surface of the structure. The relationship between sound pressure and velocity is given by [2]

$$v_f(x) = \frac{1}{i\omega \rho_f} \frac{\partial p(x)}{\partial n(x)}.$$  

(2)

At infinity the Sommerfeld radiation condition applies, which sets the sound pressure to zero at infinity at a certain decay rate [1, 2].

2.2 Discretization

The spatial sound pressure field, as described by the Helmholtz equation in Eq. (1), can be discretized and computed by application of FEM and the (mapped) conjugated Astley-Leis Infinite Element Method (IFEM) [4–6]. The domain of infinite elements is attached to an inner, circular FE domain, which discretizes the fluid around an inner obstacle. The IFEM approach is structured very similarly to the Finite Element Method. The Helmholtz equation is expressed in a variational form with the introduction of test functions. In the next step, the integral is solved by application of the product rule and the divergence theorem, which incorporates the boundary conditions. Sound pressure and structural velocity are then discretized by basis functions and the integrals are finally solved numerically in order to construct discrete system matrices of stiffness ($K$), damping ($D$) and mass ($M$). These can be found in the resulting linear system of equations

$$(K - ikD - k^2M)p = i\omega \rho_f \Theta v_s = f,$$  

(3)

where $p$ is a discrete vector of nodal pressure values, $v_s$ is the vector of nodal structural velocities and $\Theta$ is the boundary mass matrix [2]. The FE matrices are then filled with non-zero values at the rows and columns corresponding to the degrees of freedom in the domain of the infinite elements.

In order to calculate the system matrices elementwise, the infinite elements are mapped to a local Cartesian coordinate system with the coordinates $t = [-1, 1]$ in the transverse direction and $s = [-1, 1]$ in the radial direction. More details on the transformation can be found in the work of Marques and Owen [15].

For the transverse interpolation, the polynomials are inherited from the FE basis functions and therefore Lagrange polynomials of first or second order. In the radial direction, the interpolation is implemented by using either Lagrange polynomials, Legendre polynomials or Jacobi polynomials. Shirron and Babuška [7] as well as Dreyer and von Estorff [8] have shown that Lagrange polynomials lead to ill-conditioned system matrices as the polynomial degree increases and Legendre or Jacobi polynomials result in noticeably better-conditioned matrices. The authors confirmed this observation in their own studies and have shown that the choice of the type of radial interpolation polynomial has hardly any effect on normal modes and acoustic radiation modes in exterior acoustics [13]. In this paper, the authors focus on normal modes and use Jacobi ($\alpha = 1$ and $\beta = 0$) polynomials for radial interpolation, since they provide the best matrix condition.
Normal modes can be found as right eigenvectors of a state-space formulation of the quadratic eigenvalue problem in Eq. (3) that was extensively presented by Marburg et al. [10–12]

\[(A + ikB)z = r\]  \hspace{1cm} (4)

where \(r^T = [0, -f] = [0, -i\omega\rho_f\Theta v_s]\) and the hypermatrices \(A\) and \(B\) are constructed from the frequency-independent matrices of stiffness, damping and mass

\[
A = \begin{bmatrix} M & 0 \\ 0 & -K \end{bmatrix}, \quad B = \begin{bmatrix} 0 & M \\ M & D \end{bmatrix} \quad \text{and} \quad z = \begin{bmatrix} -ikp \\ p \end{bmatrix}.
\]  \hspace{1cm} (5)

The matrices \(A\) and \(B\) are asymmetric so that left and right eigenvectors \(y_{z,i}\) and \(x_{z,i}\) of the same length as the number of rows and columns of the state-space matrices are obtained (indexed with the subscript \(z\)) [11]

\[(A - \kappa_i B)x_{z,i} = 0 \quad \text{and} \quad y_{z,i}^T(A - \kappa_i B) = 0.\]  \hspace{1cm} (6)

The corresponding modal matrices \(Y\) and \(X\) diagonalize the state-space matrices \(A\) and \(B\)

\[
Y_z^TAX_z = \text{diag}(\alpha_1, \ldots, \alpha_{2N-\delta})
\]  \hspace{1cm} (7)

\[
Y_z^TBX_z = \text{diag}(\beta_1, \ldots, \beta_{2N-\delta})
\]  \hspace{1cm} (8)

such that \(\alpha_i\) and \(\beta_i\) can be found on the diagonal. The eigenvalues \(\kappa_i = \alpha_i/\beta_i\) are expected to be most accurate in the case of perfectly diagonal matrix products in Eqs. (7) and (8), whereas entries on the minor diagonals lead to a loss of information and therefore inaccurate eigenvalues \(\kappa_i\). The imaginary part of the eigenvalue denotes the resonance frequency of the corresponding mode, whereas the modal damping is given in its real part.

Finally, the eigenvectors or modal matrices are reduced to their lower \(N\) rows corresponding to the degrees of freedom associated with the sound pressure in the matrix \(z = [-ikp, p]^T\), where \(N\) is the number of rows or columns of the quadratic system matrices.

3. MODEL AND IMPLEMENTATION

The subsequent investigations are made in the example of an air-filled domain (\(c_f = 340\ \text{m s}^{-1}\) and \(\rho_f = 1.3\ \text{kg m}^{-2}\)) around the cross-section of a recorder. A circular domain of finite elements was computed with COMSOL Multiphysics and the mesh as well as the FE matrices were imported into Matlab for further calculations. The maximum element size of 3.78 cm was chosen such that the respective number of elements per wavelength gives a sufficiently fine sampling to ensure reliable results up to a frequency of 3 kHz to 4 kHz [16]. In total, the FE mesh consists of 1021 finite elements of quadratic order with 2197 degrees of freedom and has 56 outer boundary lines at the junction to the IFE domain.

The IFEM approach was implemented in Matlab and the global system matrices were constructed. The number of radial interpolation points of the infinite elements was varied from 2 up to 20. This number of interpolation points is also equal to the degree of the chosen Jacobi polynomials.

![Figure 1 – Recorder with surface nodes in the fluid.](image)

Solving the eigenvalue problem in Eq. (5) with the Matlab function \(\text{eig}(A, B)\) gives the normal modes as right eigenvectors \(x_i\) and the corresponding eigenvalues \(\kappa_i\).

4. RESULTS

4.1 Convergence of the normal mode eigenvalues

The normal mode shapes are the right eigenvectors \(x_i\) that can be found by solving the eigenvalue problem in Eq. (4), considering only the lower half of the degrees of freedom in the state-space vectors associated with
the sound pressure, see Eq. (5). Three mode shapes in the fluid around the recorder are illustrated in an image section in Fig. 2 for Jacobi(1, 0) polynomials of order 20. Their resonance peaks can mainly be observed in the interior of the hollow recorder, i.e. in the upper part of the barrel close to the mouthpiece (see Fig. 2(a)), in the lower part of the barrel (see Fig. 2(b)) and in the mouthpiece of the recorder (see Figs. 2(c) and 2(d)). The related resonance frequencies can be found in the imaginary parts of their corresponding eigenvalues, $\kappa_a$, $\kappa_b$ and $\kappa_c$, respectively. By application of the Modal Assurance Criterion (MAC) [17], these eigenvectors can be compared to those from other calculations with modified polynomial degree and the same or even similar mode shapes can be identified for different computations. For this purpose, only the degrees of freedom on the surface of the recorder were taken into consideration, since most of the mode shapes can be identified by the sound pressure distribution on the surface.

$$\begin{align*}
\kappa_a &= \kappa_{615} = -47.02 + 818.6i \\
\kappa_b &= \kappa_{990} = -94.11 + 1768.7i \\
\kappa_c &= \kappa_{1334} = -45.73 + 2861.11i
\end{align*}$$

Figure 2 – Three examples of normal mode shapes as right eigenvectors in the fluid using Jacobi(1, 0) polynomials of order 20.

In Fig. 3 the normal mode eigenvalues are depicted in the complex plane for radial interpolation polynomials of order 10 and 20, considering real parts of comparatively small magnitude ($|\Re\{\kappa_j\}| < 100$ which means small damping) and imaginary parts below 3 kHz, neglecting complex conjugates. It can be observed that the eigenvalues move in the complex plane as the number of radial interpolation points is increased in the domain of the infinite elements. In addition, the number of calculated eigenvalues increases with the number of degrees of freedom. Several of the calculated eigenvalues are expected to correspond to spurious modes [11]. At least the eigenvalues of physically meaningful eigenvectors are expected to converge as the order of the interpolation polynomials is increased. The convergence plots in Fig. 4 confirm this expectation, where the real and imaginary parts of the three considered eigenvalues are depicted as a function of radial interpolation points $n_{rad}$. For each of the three eigenvalues, two convergence curves can be found respectively for polynomials of even (blue) and odd (red) order.

$$\begin{align*}
\kappa_j \text{ at } n_{rad} = 10 \\
\kappa_j \text{ at } n_{rad} = 20
\end{align*}$$

Figure 3 – Complex plane with normal mode eigenvalues computed with radial interpolation polynomials of: (a) degree 10 and (b) degree 20 [13]. The red circles indicate the eigenvalues $\kappa_a$, $\kappa_b$ and $\kappa_c$ corresponding to the eigenvectors in Fig. 2.
4.2 Orthogonality of the normal modes

The accuracy of the calculated normal mode eigenvalues $\kappa_i = \alpha_i/\beta_i$ depends on the orthogonality of the modal matrices. These matrices diagonalize the state-space matrices such that $\alpha_i$ and $\beta_i$ can be obtained on the main diagonal as given in Eqs. (7) and (8). The ratio of the entries on the main diagonal to other values on the off-diagonals in the same rows (i) and columns (j) can be calculated with the following equation, which yields a vector of the length of rows and columns of $H = [h_{ij}]$:

$$d_H = \frac{\sum_i H^T | \sum_j H |}{2 | diag(H)|},$$

where $H = Y^T A X$ or $H = Y^T B X$. The best orthogonality is given when $d$ is equal to zero. Large values $\gg 1$ (outliers) in $d$ indicate comparatively large entries on the off-diagonals in the respective rows or columns.

The vectors of diagonal quality $d_A$ and $d_B$ are depicted in Fig. 5, where the red points show the entries of the vector $d_A$ in the upper row and the blue points show the entries of $d_B$ in the second row. From left to right, the number of radial interpolation points increases from $n_{rad} = 2$ in the first column to the calculation using $n_{rad} = 7$ in the very last column. In Table 1, the number of outliers is given for each number of radial interpolation points from two to twenty. The number of outliers increases roughly as the number of degrees of freedom grows.

Table 1 – Number of outliers $d \gg 1$ in the vectors of diagonal quality of the matrix products of the modal matrices and the state-space matrices $A$ and $B$

<table>
<thead>
<tr>
<th>$n_{rad}$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
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<th>14</th>
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<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of outliers in $d_A$</td>
<td>0</td>
<td>3</td>
<td>5</td>
<td>5</td>
<td>9</td>
<td>11</td>
<td>13</td>
<td>15</td>
<td>17</td>
<td>20</td>
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<td>24</td>
<td>23</td>
<td>27</td>
<td>29</td>
<td>32</td>
<td>31</td>
<td>36</td>
<td>36</td>
</tr>
<tr>
<td>number of outliers in $d_B$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>8</td>
<td>7</td>
<td>8</td>
<td>17</td>
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<td>21</td>
<td>18</td>
<td>19</td>
<td>19</td>
</tr>
<tr>
<td>$\Sigma$</td>
<td>1</td>
<td>5</td>
<td>8</td>
<td>9</td>
<td>14</td>
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</table>
In the subsequent section, the modal contribution of single normal modes to the radiated sound power is investigated and the influence of neglecting outliers is discussed.

4.3 Radiated sound power

A structural velocity on the surface of an obstacle leads to the radiation of sound into the surrounding fluid. The radiated sound power is an omnidirectional quantity for the description of sound sources. The discrete form of the sound power is given by [11]

\[ P = \frac{1}{2} \Re \{ \mathbf{p}_f^T \mathbf{\Theta}_f \mathbf{v}_f^* \}, \]  

(10)

where \( \mathbf{p}_f \) and \( \mathbf{v}_f \) are the discrete vectors of nodal sound pressure and structural particle velocity considering the degrees of freedom on the surface of the radiator and where \( \mathbf{\Theta}_f \) is the boundary mass matrix.

An arbitrary normal structural velocity \( \mathbf{v}_f \) is defined at the finite element nodes on the labium of the recorder. The sound power in Eq. (10) can be calculated by substitution of the vector of nodal sound pressure values \( \mathbf{p}_f \). This can be found by inverting the frequency-dependent discrete system matrix, the term in brackets in Eq. (3), so that

\[ P_{\text{inv}} = \frac{1}{2} \Re \{ (\mathbf{A}^{-1} \mathbf{f})^T \mathbf{\Theta}_f \mathbf{v}_f^* \}, \]  

(11)

where \( \mathbf{A} = (\mathbf{K} - i \omega \mathbf{D} - k^2 \mathbf{M}) \) and the subscript \( \Gamma \) means, that the resulting vector is reduced to the rows that correspond to nodes on the surface. In Fig. 6(a), the sound power is depicted as a function of frequency, where the blue curves indicate even polynomial degrees for the radial interpolation polynomials and the red curves correspond to polynomials of odd degrees. In addition, a reference solution is computed by using COMSOL Multiphysics with Perfectly Matched Layers (PML). It is depicted as a black curve in Fig. 6(a). It can be observed, that the sound power differs for interpolation polynomials of even and odd order. Both solutions converge as the polynomial degree increases.

Marburg [11] presented a method to calculate the sound power with normal modes. The computed curves are depicted in Fig. 6(b). The sound power was computed sufficiently accurately by the normal modes for radial interpolation polynomials of order 2, 9, 11, 12 and 20. The remaining calculations produced deficient curves and are denoted with the corresponding polynomial degree \( n_{\text{rad}} \). With respect to the outliers in the vector of diagonal quality, which are given in Fig. 5 and in Tab. 1, hardly any coherence can be observed between the number of outliers and the accuracy of the computed sound power. Considering the single modal contributions (depicted as black dotted lines) to the total sound power in Fig. 7, it becomes clear that the outliers from diagonality are not solely responsible for wrong results for the total superimposed sound power. In Fig. 7(a) and in Fig. 7(b) the modal contributions are depicted for \( n_{\text{rad}} = 17 \) and \( n_{\text{rad}} = 20 \) respectively. The reference solution is given as a black curve and the results obtained by fully inverting the linear system of equations according to Eq. (11) are depicted in red for the odd degree 17 and in blue for the even degree 20. The modal contributions that correspond to the outliers from diagonality of the diagonalized modal matrices are colored magenta (\( n_{\text{rad}} = 17 \)) and cyan (\( n_{\text{rad}} = 20 \)), respectively. Most of these outliers lead to a very
Figure 6 – (a) Radiated sound power calculated by full inversion of the global system matrix for polynomials of even (blue) and odd (red) order as well as reference solution by COMSOL Multiphysics (black); (b) Additionally: sound power by normal modes with dashed lines (the numbers denote the polynomial degrees with deficiently computed sound power).

low modal sound power that has hardly any influence on the total radiated sound power resulting from modal superposition, which is in turn depicted as a dashed line. However, single modal contributions lead to very large sound power curves that might be fully responsible for the deficient results.

Figure 7 – Radiated sound power for (a) \( n_{\text{rad}} = 17 \) (red and magenta) and (b) \( n_{\text{rad}} = 20 \) (blue and cyan): Reference solution (---), calculation by fully inverting the linear system of equations \( P_{\text{Ainv}} \) in Eq. (11) (-- -- -- --), modal contributions by normal modes (· · · · ·) and modal superposition neglecting the outliers from diagonality (· · · · ·). The complete modal superposition and the reduced modal superposition give almost the same curves for both cases (a) and (b).
5. CONCLUSION

Increasing the number of radial interpolation points in the domain of the infinite elements reveals two different convergences of normal mode eigenvalues, depending on whether the polynomial is of even or odd order. These two convergences were also observed for the calculated sound power by fully inverting the global system matrix containing the matrices of stiffness, damping and mass. The sound power can also be calculated from the modal matrices of the normal modes or by modal superposition of single eigenvectors. Some of the calculations result in deficient sound power curves. The diagonal quality of the matrices that provide the normal mode eigenvalues is investigated, since it is expected that entries on the off-diagonals lead to a loss of information and therefore to insufficient eigenvalues on the diagonals. The modal sound power contributions that correspond to outliers from the diagonal lead to very low modal sound power curves and seem to have a minor influence on the total modal superposition. However, single modal contributions yield comparatively large modal sound power curves over the whole frequency range to such extent that they can be expected to lead to deficient results for the total radiated power. Future work will need to clarify, which modal contributions are related to spurious modes and which are therefore non-physical mode shapes, in order to discuss the possibilities of modal reductions. In addition, the reasons for different eigenvalues for IFEM interpolation polynomials of even and odd order require further investigations. In particular, alternative eigenvalue solvers have to be taken into consideration.

REFERENCES


