On the use of a variational technique based on integral equations for plane acoustic and vibro-acoustic problems

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ABSTRACT

Problems such as sound insulation and absorption of plane structures in laboratory conditions can theoretically be described as an integral or integral-differential equation. This equation contains the Green's function integrated over the surface, which describes the radiation from the surface. A variational technique, well described by Morse and Ingard, has successfully been used for both absorption and sound insulation for a plane incident wave. The resulting formulas are surprisingly simple, accurate and robust. Moreover, they capture the physics of sound radiation of a finite surface well. However, the approach has turned out problematic in more complicated cases, including spatial periodicity. The paper discusses these issues, and suggests modifications to overcome the problems.

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1. INTRODUCTION

The variational technique for finding approximate solutions to integral equations as outlined by Morse and Ingard [1] have turned out very useful for describing coupled problems of finite extent, such as sound absorbers [2-5], sound transmission trough single walls [6], and excitation of thick plates [7]. The main strength of the method is that it usually results in very simple approximation of the problem, with a second order error. However, it have been found for spatially periodic problems will the variational technique produce poor results.[8-9] This paper goes through the theory behind the variational technique, and discusses its use as well as its pros and cons.

2. THE GENERAL THEORY

2.1 The basic integral equation and its variational equation

The integral equation considered by Mores and Ingard [1] has the following form

$$y(\mathbf{x}) = Ae^{-i\mathbf{k}\cdot\mathbf{x}} - \frac{1}{2}\int y(\mathbf{x}')\, K(\mathbf{x}'|\mathbf{x}) dS', \tag{1}$$

where we are using a slightly different notation and are considering a 2D problem, and the time convention $e^{i\omega t}$. The term $Ae^{-i\mathbf{k}\cdot\mathbf{x}}$ corresponds to an incident plane wave propagating in the positive direction, with wavenumber vector $\mathbf{k} = (k_x,\ k_y)$ and space vector $\mathbf{x} = (x,\ y)$. The second right hand term is an integral over the surface and contains a symmetric kernel $K(\mathbf{x}'|\mathbf{x}) = K(\mathbf{x}|\mathbf{x}')$, that for example can be due to the radiation of the surface as described with point source solutions, i.e., a Green's function. The integral equation is a complex valued and a Fredholm integral equation of the second kind.

The approach outlined in Mores and Ingard [1] is as follows. First, we need to consider an adjoint problem

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$$y_a(\mathbf{x}) = Ae^{i\mathbf{k}\cdot\mathbf{x}} - \frac{1}{2}\int y_a(\mathbf{x}')\, K(\mathbf{x}'|\mathbf{x})dS', \tag{2}$$

which corresponds to the same problem but with the incident wave coming in the opposite direction. The reason for considering this adjoint problem is to make it suitable for complex valued problems. Multiply with the adjoint function and integrate

$$\int y_a(\mathbf{x})y(\mathbf{x})dS = A\int y_a(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}dS - \frac{1}{2}\int\int y_a(\mathbf{x})y(\mathbf{x}')\, K(\mathbf{x}'|\mathbf{x})dS\, dS', \tag{3}$$

And tactically define the functional to be

$$V = A\int y(\mathbf{x})e^{i\mathbf{k}\cdot\mathbf{x}}dS, \tag{4}$$

which typically is related to the a power measure of the studied system. Equations (3) and (4) are now combined to form

$$\begin{aligned} V(y, y_a) = A\int y(\mathbf{x})e^{i\mathbf{k}\cdot\mathbf{x}}dS + A\int y_a(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}dS \\ - \int y_a(\mathbf{x})y(\mathbf{x})dS - \frac{1}{2}\int\int y_a(\mathbf{x})y(\mathbf{x}')\, K(\mathbf{x}'|\mathbf{x})dS\, dS', \end{aligned} \tag{5}$$

It should be noted that if instead starting with the adjoint problem and $V_a = A\int y_a(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}dS$, we will end up with the same variational formulation.

To proceed, say that the true solution to equation (1) is $y(\mathbf{x}) = Y(\mathbf{x})$. Then consider

$$y(\mathbf{x}) = Y(\mathbf{x}) + \lambda v(\mathbf{x}),\ y_a(\mathbf{x}) = Y_a(\mathbf{x}) + \lambda v^*(\mathbf{x}) \tag{6}$$

where λ is a parameter and $v(\mathbf{x})$ is an arbitrary function. Inserting these in (5) yields

$$\begin{aligned} V(y, y_a) &= V(Y, Y_a) + \lambda\int v^*(\mathbf{x})\left[Ae^{-i\mathbf{k}\cdot\mathbf{x}} - Y(\mathbf{x}) - \frac{1}{2}\int Y(\mathbf{x}')\, K(\mathbf{x}'|\mathbf{x})dS'\right]dS \\ &+\lambda\int v(\mathbf{x})\left[Ae^{i\mathbf{k}\cdot\mathbf{x}} - Y_a(\mathbf{x}) - \frac{1}{2}\int Y_a(\mathbf{x}')\, K(\mathbf{x}'|\mathbf{x})dS'\right]dS \\ &-\lambda^2\left(\int v^*(\mathbf{x})v(\mathbf{x})dS + \frac{1}{2}\int\int v(\mathbf{x}')v^*(\mathbf{x})\, K(\mathbf{x}'|\mathbf{x})dS\, dS'\right), \end{aligned} \tag{7}$$

where

$$V(Y, Y_a) = A\int Y(\mathbf{x})e^{i\mathbf{k}\cdot\mathbf{x}}dS + A\int Y_a(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}dS - \int Y_a(\mathbf{x})Y(\mathbf{x})dS - \frac{1}{2}\int\int Y_a(\mathbf{x})Y(\mathbf{x}')\, K(\mathbf{x}'|\mathbf{x})dS\, dS'.$$

In order that the functional $\partial V(y)/\partial\lambda = 0$ at $\lambda = 0$, the quantity in the two square bracket of (7) must be zero. Thus, the original two equations (1-2) must be fulfilled. This proves that the functional V is a variational functional.

The standard use of the variational formulation is to find what we can call a forced solution. Let's assume the solution to be of the same spatial form as the exciting wave

$$y(\mathbf{x}) = \lambda e^{-i\mathbf{k}\cdot\mathbf{x}},\ y_a(\mathbf{x}) = \lambda e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{8}$$

Note that the adjoint field $y_a(\mathbf{x})$ is formed with the exciting wave coming in the opposite direction, but that it has the same amplitude λ as the original field $y(\mathbf{x})$. Insert these in (5),

$$V(\lambda) = A\int \lambda e^{-i\mathbf{k}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{x}} dS + A\int \lambda e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{x}} dS - \int \lambda e^{i\mathbf{k}\cdot\mathbf{x}} \lambda e^{-i\mathbf{k}\cdot\mathbf{x}} dS - \frac{1}{2}\int\int \lambda e^{i\mathbf{k}\cdot\mathbf{x}} \lambda e^{-i\mathbf{k}\cdot\mathbf{x}'} K(\mathbf{x}'|\mathbf{x}) dS\, dS', \tag{9}$$

It should here be noted at the sign of the $e^{-i\mathbf{k}\cdot\mathbf{x}}$ and $e^{i\mathbf{k}\cdot\mathbf{x}}$ components cancel out each other, except in the last term including the kernel, and we get

$$V(\lambda) = 2A\lambda\int dS - \lambda^2\int dS - \frac{1}{2}\lambda^2\int\int e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{x}'} K(\mathbf{x}'|\mathbf{x}) dS\, dS', \tag{10}$$

where $\int dS = S$. The parameter λ is now found by finding a stationary point in *V*,

$$\frac{\partial V(\lambda)}{\partial \lambda} = 2A\int dS - 2\lambda\int dS - \lambda\int\int e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{x}'} K(\mathbf{x}'|\mathbf{x}) dS\, dS' = 0, \tag{11}$$

and thus

$$\lambda = \frac{A}{1 + \frac{1}{2S}\int\int e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{x}'} K(\mathbf{x}'|\mathbf{x}) dS\, dS'} \tag{12}$$

In equation (8) we assumed that the adjoint field had the same parameter λ as the original field, which perhaps do not seem justified. If we instead assume them to have two individual parameters,

$$y(\mathbf{x}) = \lambda e^{-i\mathbf{k}\cdot\mathbf{x}},\ y_a(\mathbf{x}) = \lambda_a e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{13}$$

equations (10) and (11) will then be replaced by

$$V(\lambda, \lambda_a) = A\lambda\int dS + A\lambda_a\int dS - \lambda_a\lambda\int dS - \frac{1}{2}\lambda_a\lambda\int\int e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{x}'} K(\mathbf{x}'|\mathbf{x}) dS\, dS', \tag{14}$$

$$\frac{\partial V(\lambda)}{\partial \lambda_a} = A\int dS - \lambda\int dS - \frac{1}{2}\lambda\int\int e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{x}'} K(\mathbf{x}'|\mathbf{x}) dS\, dS' = 0. \tag{15}$$

Thus, comparing equation (15) with (11), we see that they are equal, and (15) will also lead to equation (12).

2.2 The absorption case

The case with a finite sized absorber in an infinite baffle have been thoroughly studied by Thomasson [2,3] and other [4,5]. A simplified version of the problem (an absorbing strip) can be found already in Morse and Ingard [1], pages 458-463. The integral equation is found as a special case of the Kirchhoff-Helmholtz integral equation, considering just one plane surface,

$$p(\mathbf{r}) = \int_V G(\mathbf{r}|\mathbf{r_0}) f(\mathbf{r})\, dV + \int_S \left(p(\mathbf{r})\frac{\partial G(\mathbf{r}|\mathbf{r_0})}{\partial z} - G(\mathbf{r}|\mathbf{r_0})\frac{\partial p}{\partial z}\right) dS_0, \tag{16}$$

where $p(\mathbf{r})$ is the acoustic pressure in point $\mathbf{r} = (x, y, z)$ the sources are described by $f(\mathbf{r})$, $G(\mathbf{r}|\mathbf{r_0})$ is the Green's function. If we now consider an absorbing material with normalized impedance $z = 1/\beta$, where β is the normalized admittance, occupying a region A in the plane z=0. Outside this region, in the same plane z=0, the normalized admittance is zero – a rigid baffle. Thus, the boundary conditions are

$$\left.\frac{\partial p}{\partial z}\right|_{z=0} = \begin{cases} ik\beta p & \text{if } \mathbf{x} \in A \\ 0 & \text{if } \mathbf{x} \notin A \end{cases} \tag{17}$$

Now, by selecting the Green's function to be

$$G(\mathbf{r}|\mathbf{r_0}) = \frac{e^{-ikR_1}}{4\pi R_1} + \frac{e^{-ikR_2}}{4\pi R_2} \xrightarrow[z_0=0]{z=0} \frac{e^{-ikR}}{2\pi R} \tag{18}$$

where $R_{1,2} = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z \mp z_0)^2}$ and $R = \sqrt{(x-x_0)^2 + (y-y_0)^2}$. For incident plane waves, the source term in equation (16) simply is $\hat{p}_i\left(e^{-i\mathbf{k}\cdot\mathbf{x}}e^{-ik_z z} + e^{-i\mathbf{k}\cdot\mathbf{x}}e^{ik_z z}\right)$.

$$p(\mathbf{r}) = \hat{p}_i\left(e^{-i\mathbf{k}\cdot\mathbf{x}}e^{-ik_z z} + e^{-i\mathbf{k}\cdot\mathbf{x}}e^{ik_z z}\right) - ik\beta \int_S p(\mathbf{x}_0) G(\mathbf{r}|\mathbf{x}_0) dS_0, \tag{19}$$

which on the surface *z=0* is

$$p(\mathbf{x}) = 2\,\hat{p}_i e^{-i\mathbf{k}\cdot\mathbf{x}} - ik\beta \int_S p(\mathbf{x}_0) G(\mathbf{x}|\mathbf{x}_0) dS_0. \tag{20}$$

This is now an integral equation of the form in equation (1), and we can from equation (12) find the forced solution to be $p(\mathbf{x}) = \lambda e^{-i\mathbf{k}\cdot\mathbf{x}}$ where

$$\lambda = \frac{2\,\hat{p}_i}{1 + \frac{ik\beta}{S}\int\int e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{x}'}\, G(\mathbf{x}'|\mathbf{x}) dS\, dS'}. \tag{21}$$

Thomasson [?,?] here identifies a normalized radiation (or field) impedance

$$z_f = \frac{ik\beta}{S}\int\int e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{x}'}\, G(\mathbf{x}'|\mathbf{x}) dS\, dS', \tag{22}$$

and the pressure field at the absorber surface is estimated to be

$$p(\mathbf{x}) = \frac{2\,\hat{p}_i}{1 + \beta z_f} e^{-i\mathbf{k}\cdot\mathbf{x}} = \frac{2\,\hat{p}_i z}{z + z_f} e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{23}$$

From this result, Thomasson derive an expression for the absorption coefficient for a finite size absorber, which for one incident wave can be expressed as

$$\alpha(\mathbf{k}) = \frac{P_{abs}}{P_{inc}} = \frac{4\mathrm{Re}\{z\}/\cos\theta}{\left|z + z_f\right|^2}. \tag{24}$$

(Thomasson actually do just state the corresponding result for the random incident case, integrating over all incident angles, and do not explicitly give this intermediate result. It can however be found in [10].) With this simple formula can the area dependence (and thereby the so called "edge-effect") be described, where the radiation impedance, depending on the geometrical shape of the absorption patch and the incidence angles, is taking the finiteness into account.

2.3 The transmission case

Based on Morse and Ingard [1] and Thomasson [2,3], Brunskog [7] considered the case of transmission through a single wall, which is the simplest element of concern in building acoustics. Thus, consider a single wall of finite size is located in an infinite baffle, an acoustically hard plane, at $z = 0$. On the source side, the total acoustic field will consist of one plane incident wave p_i, one plane geometrically reflected wave p_r and one scattered field p_s that is due to the motion of the finite wall, $p = p_i + p_r + p_s$. On the receiver side, only the transmitted wave is present, $p = p_t$. The scattered and transmitted wave will be described by means of the Rayleigh integral,

$$p_{s,t}(r) = \pm i\omega\rho \int_{S_w} v(x')G(x,z|x',0)dS' \tag{25}$$

The vibrations of the wall can in general terms be describe with a differential operator

$$\mathcal{Z}v(x) = p_i(x,0) + p_r(x,0) + p_s(x\,0) - p_t(x,0) \tag{26}$$

where the wall impedance operator $\mathcal{Z}$, including both the plate and the beams, will be described in more detail below. Using the equations above,

$$\mathcal{Z}v(S) = 2\hat{p}_i e^{-ik\cdot x} + 2i\omega\rho \int_S v(x')G(x|x')dS'. \tag{27}$$

This is an integral-differential equation with the vibration velocity of the wall v as unknown. Note that the number 2 in the last term is due to radiation to both sides of the plate. Equation (27) corresponds to equation (1) with the exception for the impedance operator $\mathcal{Z}$, which is used as a general description of wall. If the wall is a thin plate, the Kirchhoff plate equation is the governing equation for the wave motion in the plate,

$$\mathcal{Z} = \mathcal{Z}_p = \frac{B'}{i\omega}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 + i\omega m''. \tag{28}$$

From Brunskog [7], the variational formulation for finite single walls results in the functional

$$V(v,v_a) = 2\hat{p}_i \int_S v(x)e^{ik\cdot x}\,dS + 2\hat{p}_i \int_S v_a(x)e^{-ik\cdot x}\,dS - \int_S v_a(x)\mathcal{Z}v(x)\,dS \\ +2i\omega\rho \int_S \int_S v_a(x)v(x')\,G(x|x')dS'\,dS, \tag{29}$$

which corresponds to equation (5), with the exception for the term containing $\mathcal{Z}$. However, use of assumed test functions of the forced type, this will be no problem, as

$$\int_S v(x)\mathcal{Z}v_a(x)\,dS = \int_S v_a(x)\mathcal{Z}v(x)\,dS \tag{30}$$

and whole variational is symmetric in v and v_a. If this is not the case, the following functional can be constructed, combining (28) with the corresponding result for the adjoint field,

$$V(v,v_a) = 2\hat{p}_i \int_S v(x)e^{ik\cdot x}\,dS + 2\hat{p}_i \int_S v_a(x)e^{-ik\cdot x}\,dS \\ -\tfrac{1}{2}\int_S v_a(x)\mathcal{Z}v(x)\,dS - \tfrac{1}{2}\int_S v(x)\mathcal{Z}v_a(x)\,dS + 2i\omega\rho \int_S \int_S v_a(x)v(x')\,G(x|x')dS'\,dS. \tag{31}$$

Following the same procedure as equations (8) to (12), the forced solution is

$$v(x) = \frac{2\hat{p}_i}{Z_p + 2\rho c\, z_f} e^{-ik\cdot x}, \qquad \tau = \frac{\omega\rho^2 c^2 \Re\{z_f\}}{\cos\theta\, \left|Z_p + 2\rho c\, z_f\right|^2}. \tag{32}$$

where the second part is a simple formula for the transmission coefficient for a finite wall, for more details see [7].

2.4 Periodicity

In two previous papers [8,9] have the problem with spatial periodicity in the operator $\mathcal{Z}$ been addressed. The special periodicity can be due to beam stiffeners attached to the plate. The impedance operator for one Euler beam located in the y-direction is,

$$\mathcal{Z}_f = \frac{B_f}{i\omega}\frac{\partial^4}{\partial y^2} + i\omega m_f'. \tag{33}$$

Assume the beams to be located at $x = nl$, n being an integer and l the distance between the rib stiffeners. Also assume the connection between the beams and the plate to be in the form of a line force and equal displacement of the beam and the plate at $x = nl$. It should be noted that this is a rough simplification of the actual situation, ignoring effects as moment coupling, in-plane wave motion and eccentricity of the beams, and point vice coupling. Equation (7) can with these assumptions be written

$$\left(\mathcal{Z}_p + \sum_{n=-\infty}^{\infty} \mathcal{Z}_f \delta(x - nl)\right) v(x,y) = p(x,y) \tag{34}$$

where $p(x,y)$ is the left hand side of equation (27). The wall impedance operator $\mathcal{Z}$ can be identified as

$$\mathcal{Z} = \mathcal{Z}_p + \sum_{n=-\infty}^{\infty} \mathcal{Z}_f \delta(x - nl). \tag{35}$$

Applying this operator in equations (28-29) and (31) results in simple equations similar to (32), see [8,9]. The result is

$$\mathrm{v(x)} = \frac{2\hat{\mathrm{p}}_\mathrm{i}}{\mathrm{Z_p} + Z_b/l_x + 2\rho c\, \mathrm{z_f}} \mathrm{e}^{-\mathrm{ik \cdot x}}, \qquad \tau = \frac{\omega \rho^2 c^2 \Re\{\mathrm{z_f}\}}{\cos\theta \left|\mathrm{Z_p} + \frac{Z_b}{l_x} + 2\rho c\, \mathrm{z_f}\right|^2}. \tag{36}$$

In Figure 1 are this theory tested against experimental data, and the fit is really poor. The material is 13 mm plaster boards, supported by wooden beams of size 2” times 4”, and the distance between the beams is 40 mm. It can be seen that the line connection theory is much overestimating the sound transmission. This theory can be regarded an orthotropic theory, which will result in a spread out coincidence frequency depending of the azimuth angle. This is explaining the low TL from 200 Hz and upwards. This is not seen in the measured result, indicating that the assumptions in the line connection theory are not correct. If instead including only the mass of the beams, the TL is overestimated. The best fit is found if comparing the theory not including the beams with the measured result, indicating that the coupling between the plate and the beam is minor.

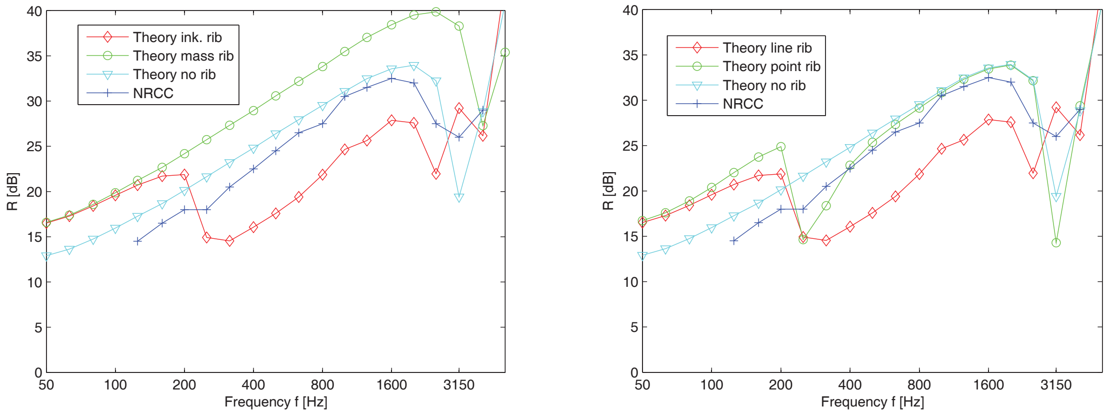

Figure 1 – Transmission loss R for plaster board wall with rib stiffeners. For both left and right: measured results by Northwood [11] (blue line + symbol); line connection theory [8] (red line diamond symbol); theory including no beams [6] (cyan line triangle symbol). Left hand figure: theory with only mass impedance of the beams (green line circle symbol). Right hand figure: point connection theory [9] (green line circle symbol).

The result of a point connection theory, as developed in [9], is also shown in Figure 1, right hand. It can be seen that, for the chosen stiffness, the theory corresponds to the unstiffened theory in the high frequency region and to the line connected theory at low frequencies. The value of the stiffness is chosen to $s = 8 \cdot 10^5$ N/m; this value is chosen to give a reasonable fit with the experimental data. The fit to the measured data is good for frequencies above 400 Hz, but the transition between the high and low frequency theories seems to be too sharp.

From these results can it be concluded the variational technique with the assumption of a forced wave field of the spatial form $\hat{v}\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$ is not suited for spatially periodic structures. In fact, analyzing the term in equation () containing the wall impedance,

$$\int_S v_a(\mathbf{x}) Z v(\mathbf{x})\, dS = \hat{v}^2 \int_S e^{i\mathbf{k}\cdot\mathbf{x}} Z e^{-i\mathbf{k}\cdot\mathbf{x}}\, dS \tag{37}$$

$$= \hat{v}^2 \int_S e^{i\mathbf{k}\cdot\mathbf{x}} Z_p e^{-i\mathbf{k}\cdot\mathbf{x}}\, dS + \hat{v}^2 \int_S e^{i\mathbf{k}\cdot\mathbf{x}} \sum_{n=-\infty}^{\infty} Z_f \delta(x - nl)\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, dS$$

$$= \hat{v}^2 Z_p \int_S dS + \hat{v}^2 Z_f \int_S \sum_{n=-\infty}^{\infty} \delta(x - nl)\, dS,$$

we see that the adjoint field cancel out the influence of the original field, and the specific information regarding the position of the beam is missing. Another way to see this is that forced wave field of the spatial form $\hat{v}\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$ is by itself fulfilling Floquet's periodic principle. Floquet's periodic principle is stating that $v(x - ml_x) = v(x)e^{imk_x l_x}$. But for $v(x) = \hat{v}e^{-ik_x x}$

$$v(x - ml_x) = \hat{v}e^{-ik_x(x - ml_x)} = \hat{v}e^{-ik_x x}e^{ik_x m l_x} = v(x)e^{imk_x l_x}. \tag{38}$$

Thus, the problem is that this periodic relationship is automatically fulfilled for the assumed vibration field. As a consequence, spatially periodic effects can never be expressed with just a forced solution.

In [8] was a sketch of a theory including the periodicity included (but never numerically evaluated). The theory turned out too complicated for a resulting simple formula as equation (31).

2.5 Serial expansion

An alternative is to use the variational functional in equation (5) as the basis of a numerical procedure, base on a series expansion

$$y(\mathbf{x}) = \textstyle\sum_n \hat{y}_n N_n(\mathbf{x}), \quad y_a(\mathbf{x}) = \sum_n \hat{y}_n^{(a)} N_n(\mathbf{x}). \tag{39}$$

The functional V is now expressed as

$$V = A \sum_n \hat{y}_n \int N_n(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} dS + A \sum_n \hat{y}_n^{(a)} \int N_n(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} dS \tag{40}$$

$$- \sum_n \sum_m \hat{y}_n^{(a)} \hat{y}_m \int N_n(\mathbf{x}) N_m(\mathbf{x}) dS - \frac{1}{2} \sum_n \sum_m \hat{y}_n^{(a)} \hat{y}_m \int\int N_n(\mathbf{x}) N_m(\mathbf{x}')\, K(\mathbf{x}'|\mathbf{x}) dS\, dS',$$

and by considering $\partial V / \partial \hat{y}_l^{(a)} = 0$, we gets

$$A \int N_i(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} dS - \sum_m \hat{y}_m \int N_m(\mathbf{x}) N_l(\mathbf{x}) dS - \frac{1}{2} \sum_m \hat{y}_m \int\int N_l(\mathbf{x}) N_m(\mathbf{x}')\, K(\mathbf{x}'|\mathbf{x}) dS dS' = 0. \tag{41}$$

This will form a system of equations, which can be expressed as a matrix equation,

$$\begin{bmatrix} \int N_1(\mathbf{x})N_1(\mathbf{x})dS + \frac{1}{2}\int\int N_1(\mathbf{x})N_1(\mathbf{x}')\,K(\mathbf{x}'|\mathbf{x})dSdS' & \cdots & \int N_1(\mathbf{x})N_M(\mathbf{x})dS + \frac{1}{2}\int\int N_1(\mathbf{x})N_M(\mathbf{x}')\,K(\mathbf{x}'|\mathbf{x})dSdS' \\ \vdots & \ddots & \vdots \\ \int N_M(\mathbf{x})N_1(\mathbf{x})dS + \frac{1}{2}\int\int N_M(\mathbf{x})N_1(\mathbf{x}')\,K(\mathbf{x}'|\mathbf{x})dSdS' & \cdots & \int N_M(\mathbf{x})N_M(\mathbf{x})dS + \frac{1}{2}\int\int N_M(\mathbf{x})N_M(\mathbf{x}')\,K(\mathbf{x}'|\mathbf{x})dSdS' \end{bmatrix} \begin{bmatrix} \hat{y}_1 \\ \vdots \\ \hat{y}_N \end{bmatrix} = A \begin{bmatrix} \int N_1(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}dS \\ \vdots \\ \int N_M(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}dS \end{bmatrix} \quad (42)$$

The intension is to investigate the accuracy of this approach, as well as the forced wave field approach with comparison with BEM calculations, but no result of this comparison is available at the moment.

3. CONCLUSIONS

The variational technique for integral equations by Morse and Ingard is a very powerful technique to reach simplified but robust formulas, taking finiteness into account for absorption and transmission, but in its simplest form, it fails to take spatial periodicity into account. A numerical procedure based on it can be developed, on the expenses of simplicity.

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