Theoretical model of scattering from serrated flat plates

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ABSTRACT
In this paper we present a theoretical model to study sound scattering from a serrated flat plat, which is of both scientific and practical importance. The key contribution of our work is the analytical and rigorous description of the scattering effect from lateral deployed serrations by incorporating Fourier series expansion and Wiener-Hopf method. A closed-form solution is then obtained by using standard routines of Wiener-Hopf factorization. The proposed model could be helpful to understand owl's silent flying mechanisms as well as to develop new low noise flying system by using serrations.

Keywords: Scattering, Serrated plates, Wiener-Hopf method

1. INTRODUCTION

In this paper, we propose a theoretical model for studying sound scattering from an idealised airfoil (a semi-infinite flat rigid plate) but with serrated open-ends (see figure 1). The sawtooth-shaped edges, which are formally known as serrations, recently become a hot topic in the area of aeroacoustics for the connection with owl's silent flying capabilities [1]. The current work is primarily focused on the scattering effect related to the serrations, which will possibly excite multiple scatterings of new modes in the lateral direction. Some of these scattered waves could be prone to cancel each other out and eventually might collectively attenuate acoustic scattering. According to our best knowledge, a theoretical model that illuminates the acoustic scattering from a serrated-plate is still not available. The current work therefore endervours to develop such a model. The key contribution is the analytical description of the acoustic scattering effect of serrations by incorporating Fourier series expansion and the Wiener-Hopf method.

The Wiener-Hopf method is a powerful mathematical tool that could yield closed-form analytical solutions for classical scattering problems such as the scattered field from a semi-infinite rigid flat plate [2]. This classical model is further extensively extended in this work to take account of the serrated open-ends. In particular, we describe periodical variations of the serrations by using Fourier series that accurately represent the geometrical layout of serrations in the lateral direction. Then, a Wiener-Hopf based theoretical model will be established to analytically examine the scattered sound field from the serrated plate. The associated matrix Wiener-Hopf kernel will be given and a rigorous closed-form solution can be obtained. Some primitive numerical tests are being performed to validate the proposed theoretical model.

2. THE THEORETICAL MODEL

2.1 Statement of the problem

As shown in figure 1, the problem considered in this work contains periodical serrations in the lateral direction. The rest semi-infinite surface is rigid and infinitely thin. For simplicity, a stationary background flow is assumed. Nevertheless, we should say that the following theoretical model can be directly applicable to cases with uniform background flows by using Prandtl-Glauert transformation. The governing equation is:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi-\frac{\partial^2}{\partial t^2}\phi=0, \tag{1}$$

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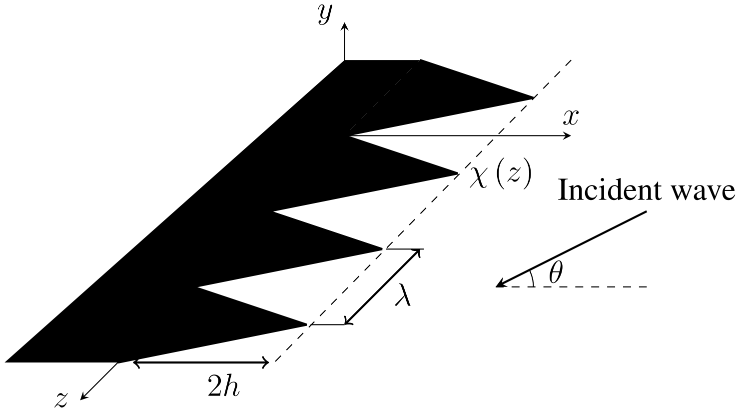


Figure 1. Sketch of the model problem.

where ϕ is the acoustic potential, and all variables are nondimensionalised using the appropriate scales, such as the mean flow density ρ_0 and the speed of sound c_0.

From ϕ, we would be able to obtain sound pressure p, particle velocity $\vec{v} = (u, v, w)$ and acoustic density ρ inside the duct by using the following formulations:

$$p = p_i + p_s = -\frac{\partial \phi}{\partial t}, \quad \vec{v} = \nabla \phi, \quad \rho = p, \tag{2}$$

where the subscripts $(\cdot)_i$ and $(\cdot)_s$ denote incident and scattering waves, respectively. The incident wave is presumably of the time-harmonic form,

$$\phi_i(x, r, \theta, t) = \psi_i(x, r, t) e^{-\ss \omega t} = A_i e^{-ik(x\cos\theta + y\sin\theta) - i\omega t}, \tag{3}$$

where A_i is the amplitude of the incident wave, $k = \omega / c_0$ is the normalised wavenumber and θ is the incident angle (see figure 1). It is easy to see that the scattered field will also takes the harmonic form,

$$\phi_s(x, y, z, t) = \psi_s(x, y, z) e^{-i\omega t}. \tag{4}$$

For simplicity, the common factor $\exp(-i\omega t)$ will be suppressed throughout the rest of this paper. Then, ψ_s should satisfy

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \psi_s + k^2 \psi_s = 0 \tag{5}$$

and subjects to the following boundary conditions:

1. On the rigid plate (where $v = 0$), the scattered field is equal but opposite to the incident field, resulting in:
$$\left. \frac{\partial \psi_s(x, y, z)}{\partial y} \right|_{y=0} = - \left. \frac{\partial \psi_i(x, y, z)}{\partial y} \right|_{y=0} = ik \sin\theta \mathrm{e}^{-ik\cos\theta x}, \forall x \leqslant \chi(z). \tag{6}$$

2. ψ_s should be an odd function in y and continuous for $x > \chi(z)$, hence
$$\left. \psi_s(x, y, z) \right|_{y=0} = 0, \forall x > \chi(z). \tag{7}$$

3. The edge condition

   The theoretical model is developed with leading edge serrations in mind. Hence, the leading edge condition is imposed at the serrated open-ends to ensure a finite velocity at the edge and to obtain a unique solution of the problem.

2.2 The Wiener-Hopf equation

We should say that the whole theoretical development of our model is not so straightforward. The main difficulty is to represent the current problem of interest in the framework of a Wiener-Hopf model with required analytical properties. The essential concept behind this model is the incorporation Fourier series into the Wiener-Hopf method. The same concept has been successfully applied in one of our recent work for sound propagations insider a cylindrical flow duct with rigid splices [3].

Now let us turn to the theoretical developments by incorporating Fourier series and Wiener-Hopf method. The latter one would enable us to obtain rigorous solutions of wave equations with a pair of idealised simple but mixed boundary conditions. Typically, the first step of Wiener-Hopf method is to transform wave equations into Wiener-Hopf equations via Fourier transform. The next step is to decompose the associate Wiener-Hopf kernel using suitable factorization methods, which should yield decomposition results analytic and bounded on upper and lower half complex planes, respectively. Then, a closed-form solution would be available by applying Liouville's theorem.

It is easy to know that ψ_s is periodic in the z-direction (with the period λ) and therefore can be represented as follows by using the Fourier series:

$$\psi_s(x,y,z) = \sum_{\kappa=-\infty}^{+\infty} \psi_s^\kappa(x,y) e^{i\kappa \frac{2\pi}{\lambda} z}, \quad \psi_s^\kappa(x,y) = \frac{1}{\lambda}\int_0^\lambda \psi_s(x,y,z) e^{-i\kappa\frac{2\pi}{\lambda}z} \mathrm{d}z, \tag{8}$$

where $(\cdot)^\kappa$ denotes the κ-th components.

Substituting (8) into (5), we would have

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2 - \left(\kappa\frac{2\pi}{\lambda}\right)^2\right] \psi_s^\kappa(x,y) = 0. \tag{9}$$

Then, we apply the conventional Fourier transform to the scattered acoustic field in the x-direction:

$$\beta(\alpha,y,z) \triangleq \int_{-\infty}^{+\infty} \psi_s(x,y,z) e^{i\alpha x}\mathrm{d}x = \int_{-\infty}^{+\infty}\left(\sum_{\kappa=-\infty}^{+\infty} \psi_s^\kappa(x,y) e^{i\kappa\frac{2\pi}{\lambda}z}\right) e^{i\alpha x}\mathrm{d}x, \tag{10}$$

where α is the axial wavenumber. It is easy to see that $\beta(\alpha,y,z)$ is also periodic in the z-direction and, therefore, can be represented as the following Fourier series and have the following relations:

$$\beta(u,y,z) = \sum_{\kappa=-\infty}^{+\infty} \beta^\kappa(u,y) e^{i\kappa\frac{2\pi}{\lambda}z}, \quad \beta^\kappa(u,y) = \int_{-\infty}^{+\infty} \psi_s^\kappa(x,y) e^{i\alpha x}\mathrm{d}x. \tag{11}$$

The latter relation assumes that the order of the integral and the summation is interchangeable.

From (11), (9) can be transferred to

$$\frac{\partial^2}{\partial y^2} - \left[\alpha^2 - \left(k^2 - (\kappa 2\pi/\lambda)^2\right)\right] \beta^\kappa(x,y) = 0. \tag{12}$$

To simplify the following formulations, we define ${\gamma^\kappa}^2 \triangleq \alpha^2 - \hat{k^\kappa}^2, \quad \hat{k^\kappa}^2 \triangleq k^2 - (\kappa 2\pi/\lambda)^2$. Then, the solutions of (12) should have the following generic form:

$$\beta^\kappa(\alpha,y) = \begin{cases} A^\kappa(\alpha) e^{-\gamma^\kappa y}, & \forall y > 0, \\ -A^\kappa(\alpha) e^{\gamma^\kappa y}, & \forall y < 0, \end{cases} \tag{13}$$

where $A^\kappa(\alpha)$ is the corresponding amplitude of the κ-th component. From (13), we can further have the following relation at $y = 0^+$:

$$\gamma^\kappa \beta^\kappa(\alpha, 0^+) + {\beta'}^\kappa(\alpha, 0^+) = 0, \forall\kappa. \tag{14}$$

where $(\cdot)'$ represents $\mathrm{d}/\mathrm{d}y$.

Now we turn to the key part of this theoretical development. First, we decompose β to the sum of the following two parts that are analytic on the two half-planes, respectively:

$$\begin{aligned}
\beta(\alpha,y,z) &= e^{i\alpha\chi(z)}\left(\int_{\chi(z)}^{+\infty}\psi_s(x,y,z)e^{i\alpha x-i\alpha\chi(z)}\mathrm{d}x+\int_{-\infty}^{\chi(z)}\psi_s(x,y,z)e^{i\alpha x-i\alpha\chi(z)}\mathrm{d}x\right)\\
&= e^{i\alpha\chi(z)}\left(\int_{0}^{+\infty}\psi_s(\hat{x}+\chi(z),y,z)e^{i\alpha\hat{x}}\mathrm{d}\hat{x}+\int_{-\infty}^{0}\psi_s(\hat{x}+\chi(z),y,z)e^{i\alpha\hat{x}}\mathrm{d}\hat{x}\right)\\
&\triangleq e^{i\alpha\chi(z)}\left[\beta_+(\alpha,y,z)+\beta_-(\alpha,y,z)\right]. \qquad (15)
\end{aligned}$$

The above definition is one of the most key steps in this theoretical model. The decomposed $\beta_\pm$ are regular on the two overlapped half-planes, which will be defined later in this paper. It is easy to see that $\exp(i\alpha\chi)$, β_+ and β_- in (15) are all periodic in the z-direction with the same period λ. By expanding them into the Fourier series, we immediately have

$$\beta(\alpha,y,z)=\sum_{\kappa_1=-\infty}^{+\infty}\zeta^{\kappa_1}e^{i\kappa_1\frac{2\pi}{\lambda}z}\sum_{\kappa_2=-\infty}^{+\infty}\left({\beta_+}^{\kappa_2}(\alpha,y,z)+{\beta_-}^{\kappa_2}(\alpha,y,z)\right)e^{i\kappa_2\frac{2\pi}{\lambda}z}, \qquad (16)$$

where

$$\begin{aligned}
e^{i\alpha\chi(z)} &= \sum_{\kappa=-\infty}^{+\infty}\zeta^{\kappa}(\alpha)e^{i\kappa\frac{2\pi}{\lambda}z},\quad \zeta^{\kappa}(\alpha)=\frac{1}{\lambda}\int_0^{\lambda}e^{i\alpha\chi(z)}e^{-i\kappa\frac{2\pi}{\lambda}z}\mathrm{d}z, \qquad (17)\\
\beta_\pm(\alpha,y,z) &= \sum_{\kappa=-\infty}^{+\infty}\beta_\pm^{\kappa}(\alpha,y)e^{i\kappa\frac{2\pi}{\lambda}z}. \qquad (18)
\end{aligned}$$

From (18), we can define the partial derivatives for the κ-th component:

$$\beta'_\pm(\alpha,y,z)=\sum_{\kappa=-\infty}^{+\infty}{\beta'_\pm}^{\kappa}(\alpha,y)e^{i\kappa\frac{2\pi}{\lambda}z}. \qquad (19)$$

Furthermore, by using the boundary condition (6), we have the analytical representation of ${\beta_-}'(\alpha,0^+,z)$:

$$\begin{aligned}
{\beta_-}'(\alpha,0^+,z) &= \int_{-\infty}^{0}{\psi_s}'(\hat{x}+\chi(z),0^+,z)e^{i\alpha\hat{x}}\mathrm{d}\hat{x}=\int_{-\infty}^{0}ik\sin\theta\mathrm{e}^{-ik\cos\theta(\hat{x}+\chi)}e^{i\alpha\hat{x}}\mathrm{d}\hat{x}\\
&= \frac{k\sin\theta}{\alpha-k\cos\theta}e^{-ik\cos\theta\chi(z)}=\frac{k\sin\theta}{\alpha-k\cos\theta}\sum_{\kappa=-\infty}^{+\infty}\upsilon^{\kappa}e^{i\kappa\frac{2\pi}{\lambda}z}, \qquad (20)
\end{aligned}$$

with the Fourier series coefficients

$$\upsilon^{\kappa}=\frac{1}{\lambda}\int_0^{\lambda}e^{-ik\cos\theta\chi(z)}e^{-i\kappa\frac{2\pi}{\lambda}z}\mathrm{d}z, \qquad (21)$$

which mathematically represents the effect of different shape of the serrations. Finally, by comparing (19) and (20), it is easy to see that

$${\beta'_-}^{\kappa}(\alpha,0^+)=\frac{k\sin\theta}{\alpha-k\cos\theta}\upsilon^{\kappa}. \qquad (22)$$

Next, by equating those terms of the same exponential order in (16) and (11), we have the following relation:

$$\begin{aligned}
\beta^{\kappa}(\alpha,y) &= \cdots+\zeta^{1}({\beta_+}^{\kappa-1}+{\beta_-}^{\kappa-1})+\zeta^{0}({\beta_+}^{\kappa}+{\beta_-}^{\kappa})+\zeta^{-1}({\beta_+}^{\kappa+1}+{\beta_-}^{\kappa+1})+\cdots\\
&= \sum_{\kappa_2=-\infty}^{+\infty}\zeta^{\kappa_2}\left({\beta_+}^{\kappa-\kappa_2}(\alpha,y)+{\beta_-}^{\kappa-\kappa_2}(\alpha,y)\right). \qquad (23)
\end{aligned}$$

By applying boundary conditions at $y=0^+$, i.e. (7) and (22), we have

$$\begin{aligned}
\beta^{\kappa}(\alpha,0^+) &= \sum_{\kappa_2=-\infty}^{+\infty}\zeta^{\kappa_2}{\beta_-}^{\kappa-\kappa_2}(\alpha,0^+), \qquad (24)\\
\beta'^{\kappa}(\alpha,0^+) &= \sum_{\kappa_2=-\infty}^{+\infty}\zeta^{\kappa_2}\left({\beta_+}'^{\kappa-\kappa_2}(\alpha,0^+)+\frac{k\sin\theta}{\alpha-k\cos\theta}\upsilon^{\kappa-\kappa_2}\right), \qquad (25)
\end{aligned}$$

where $\beta_-(\alpha, 0^+)$ is always zero due to (7). For brevity, $(\alpha, 0^+)$ will be omitted in the rest of this paper in case that no confusion will arise.

Subjecting (24)-(25) into (14) results in

$$\gamma^\kappa \sum_{\kappa_2=-\infty}^{+\infty} \zeta^{\kappa_2} {\beta_-}^{\kappa-\kappa_2} + \sum_{\kappa_2=-\infty}^{+\infty} \zeta^{\kappa_2} {\beta_+}'^{\kappa-\kappa_2} = -\frac{k\sin\theta}{\alpha - k\cos\theta} \sum_{\kappa_2=-\infty}^{+\infty} \zeta^{\kappa_2} \upsilon^{\kappa-\kappa_2}. \tag{26}$$

By repeating (26) for all possible κ, we have a system of infinite linear equations

$$\Gamma\Xi\Phi_- + \Xi{\Phi_+}' = \frac{-k\mathrm{sin}\theta}{\alpha - k\mathrm{cos}\theta}\Xi\Upsilon, \tag{27}$$

where

$$\Gamma = \mathrm{diag}(\gamma^\kappa) = \mathrm{diag}\left(\sqrt{\alpha + \hat{k^\kappa}}\sqrt{\alpha - \hat{k^\kappa}}\right) \triangleq \Gamma_+\Gamma_-, \tag{28}$$

$$\Xi = \begin{bmatrix} & & & \cdots & & & \\ \cdots & \zeta^1 & \zeta^0 & \zeta^{-1} & \cdots & & \\ & \cdots & \zeta^1 & \zeta^0 & \zeta^{-1} & \cdots & \\ & & \cdots & \zeta^1 & \zeta^0 & \zeta^{-1} & \cdots \\ & & & \cdots & & & \end{bmatrix}, \tag{29}$$

$$\Phi_- = [\cdots, {\beta_-}^{\kappa-1}, {\beta_-}^{\kappa}, {\beta_-}^{\kappa+1}, \cdots]^T, \tag{30}$$

$${\Phi_+}' = [\cdots, {\beta_+}^{\kappa-1\prime}, {\beta_+}^{\kappa\prime}, {\beta_+}^{\kappa+1\prime}, \cdots]^T, \tag{31}$$

$$\Upsilon = [\cdots, \upsilon^{\kappa-1}, \upsilon^{\kappa}, \upsilon^{\kappa+1}, \cdots]^T, \tag{32}$$

where $(\cdot)^T$ denotes the transpose operation. (27) can be further simplified to:

$$\Xi^{-1}\Gamma\Xi\Phi_- + {\Phi_+}' = \frac{-k\mathrm{sin}\theta}{\alpha - k\mathrm{cos}\theta}\Upsilon. \tag{33}$$

Nevertheless, Ξ may have a number of unknown poles at α_j. To cancel out these poles, $[\Pi(\alpha-\alpha_j)]^{-1}\Pi(\alpha-\alpha_j)$ is multiplied to the both hands of (33), resulting in

$$\underbrace{[\Xi^{-1}\Pi(\alpha-\alpha_j)^{-1}]}_{\hat{\Xi}^{-1}}\Gamma\underbrace{[\Pi(\alpha-\alpha_j)\Xi]}_{\hat{\Xi}}\Phi_- + {\Phi_+}' = \frac{-k\mathrm{sin}\theta}{\alpha - k\mathrm{cos}\theta}\Upsilon, \tag{34}$$

where Π represents the product of a sequence of $(\alpha - \alpha_j)$. Then, it is easy to see that $\hat{\Xi}$ and $\hat{\Xi}^{-1}$ in (34) are analytic. Since ${\Phi_+}'$ is still unknown, (34) is solved by using the Wiener-Hopf method. First, Γ is a diagonal matrix and its multiplicative factorisation that satisfies $\Gamma = \Gamma_+\Gamma_-$ can be easily obtained. Then,

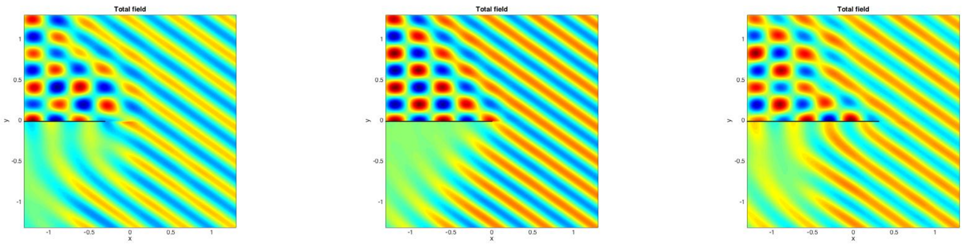
$$\Gamma_-(\alpha)\hat{\Xi}\Phi_- + \Gamma_+^{-1}(\alpha)\hat{\Xi}{\Phi_+}' = \frac{-k\mathrm{sin}\theta}{\alpha - k\mathrm{cos}\theta}\Gamma_+^{-1}(\alpha)\hat{\Xi}\Upsilon. \tag{35}$$

By re-arranging terms and canceling out the pole at $k\cos\theta$, we finally achieve the following Wiener-Hopf equation:

$$\underbrace{\Gamma_-(\alpha)\hat{\Xi}\Phi_- + \frac{k\mathrm{sin}\theta}{\alpha - k\mathrm{cos}\theta}\Gamma_+^{-1}(k\mathrm{cos}\theta)\hat{\Xi}\Upsilon}_{R_-} =$$

$$- \underbrace{\Gamma_+^{-1}(\alpha){\Phi_+}' - \frac{k\mathrm{sin}\theta}{\alpha - k\mathrm{cos}\theta}[\Gamma_+^{-1}(\alpha) - \Gamma_+^{-1}(k\mathrm{cos}\theta)]\hat{\Xi}\Upsilon}_{R_+} \equiv \mathbf{E}(\alpha). \tag{36}$$

To this end, the left- and right-hand side of (36) should be regular on the positive and negative half-planes $R_\pm$, respectively. R_+ is the positive half-plane above $\hat{k^\kappa}$, and R_- is the negative half-plane below $k\cos\theta$. As a result, according to analytical continuation in the complex analysis theory, $\mathbf{E}(\alpha)$ is en entire function



(a) (b) (c)

Figure 2: $\psi_s(x, y, z)$ of three different cross-sections passing through: (a) the root, (b) $\chi(z) = 0$ and (c) the tip. The countour levels are between ± 2.5.

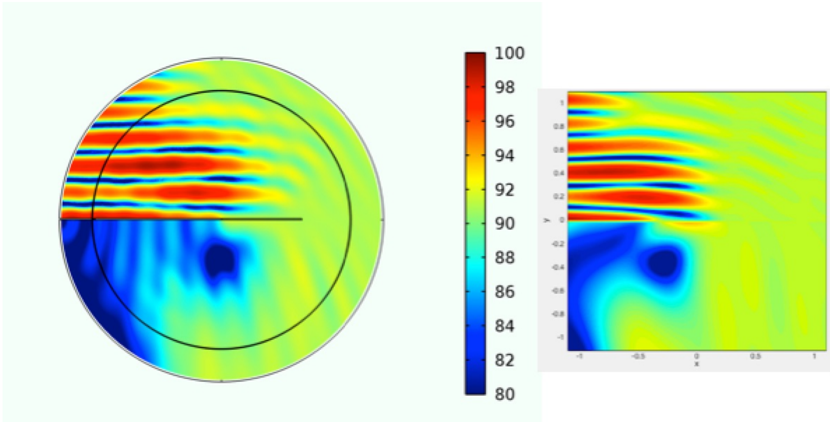


Figure 3: Comparison of the numerical (left) and analytical near-field sound pressure results (right).

that is regular on the whole complex α plane. Considering the classical case with a flat rigid plate, it will suggest that $\mathbf{E}(\alpha) \equiv 0$ as $|\alpha| \to \infty$. By subjecting this relation into (36), we finally have:

$$\Phi_-(\alpha) = -\hat{\Xi}^{-1}(\alpha)\Gamma_-^{-1}(\alpha)\frac{k\sin\theta}{\alpha - k\cos\theta}\Gamma_+^{-1}(k\cos\theta)\hat{\Xi}(\alpha)\Upsilon, \tag{37}$$

where all the right-hand side terms are known, provided the profiles of serrations and the incident wave information. We therefore have the solution of $\Phi_-(\alpha)$, and corresponding solutions for $\beta_-^{\kappa}(\alpha, 0^+)$ (using (30)), $\beta^{\kappa}(\alpha, 0^+)$ (using (24)) and $\beta^{\kappa}(\alpha, y)$ (using (13)).

Finally, we can have the scattered field:

$$\psi_s(x, y, z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\sum_{\kappa=-\infty}^{\infty}\beta^{\kappa}(\alpha, y)e^{-i\kappa\frac{2\pi}{\lambda}z}\right)e^{-i\alpha x}\mathrm{d}x. \tag{38}$$

3. RESULTS AND DISCUSSION

Here we choose one set-up to demonstrate the proposed model. The frequency of the incident wave is 1000 Hz and the incident angle is 55 deg. The profile of the serrations is $\chi(z) = h_s\sin(z/\lambda) + h_s$, where h_s is set to 0.31 and λ is set to 0.49. Figure 2 shows the corresponding near-field results at three representative cross-sectional domains. It can be seen that the near-field results are directly depend on the profile of the serrations and therefore are different from the results for classical flat plates with no serrations. We should say that the theoretical model requires the Fourier series of infinite orders and the implementation can only use truncated orders. For the current set-up, a truncation with κ in between ± 10 is already enough. In addition, the model is quite efficient in terms of memory and computational requirements.

To verify and validate the proposed model, we conduct simulations for the same set-up by using COMSOL. Figure 3 shows that the near-field sound pressure results from the numerical solver and our analytical model are quite similar at the root cross-section, in terms of patterns and amplitudes. It should be noted

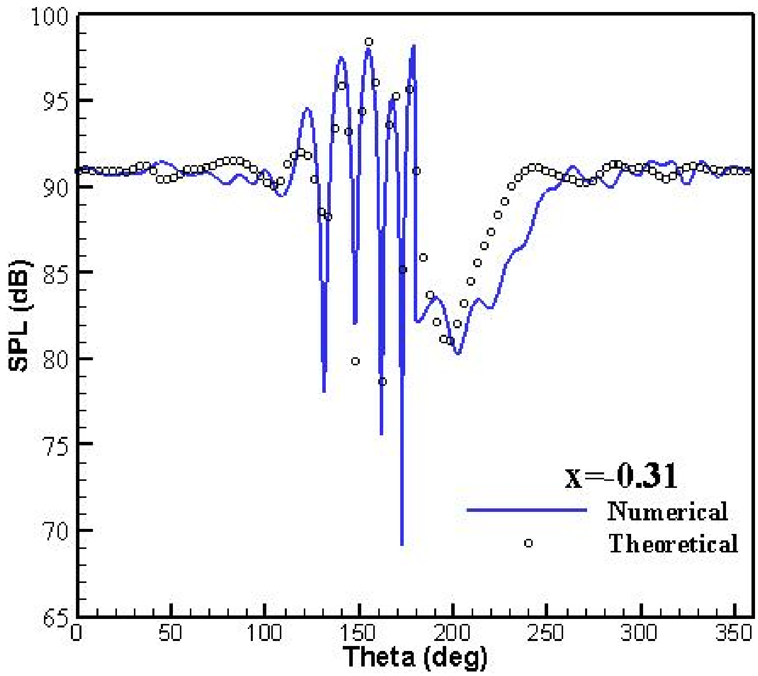


Figure 4: Comparison of the numerical and analytical results across the circle with $\sqrt{x^2+y^2} = 1$.

that the computational domain of the current numerical set-up can be further enlarged to reduce the invisible reflections. In addition, figure 3 shows a very distinctive silent region underneath the serrations.

Finally, figure 4 compares the numerical and analytical results across the circle (see figure 3) with $\sqrt{x^2+y^2} = 1$. Largely speaking, the both curves agree quite well, except in the region with θ between 180 and 260 deg. By comparing results shown in figure 3, we believe that this discrepancy is mainly caused by the spurious reflection from the numerical computational boundary. Overall, results in figures 3-4 verify and demonstrate the proposed theoretical model, and the current model can be used to study and predict possible noise attenuation effects from serrations.

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