The Reflection Equivalence Formulation for a circular ANC System

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ABSTRACT
An active noise control system (ANC) can be regarded as a sound soft reflector. Such a two-dimensional system consists of an outer microphone array (reference sensors) to detect a given sound field (primary field) and an inner loudspeaker array (secondary sources) to synthesize a secondary field, which is to eliminate the primary field inside the loudspeaker array. Thus, for a perfectly working ANC system the sound pressure will be zero along the array of secondary sources. Therefore, assuming omnidirectional secondary sources, the primary field will be reflected as if the loudspeaker array was a sound soft termination. From this point of view it is possible to determine appropriate filter functions for a circular ANC system, which will be shown in the remainder of this text. In comparison to a former formulation of the problem, the reflection equivalence allows calculating the system’s transfer functions in a more direct way and the amount of reference sensors can be reduced to the half.

Keywords: Active Noise Control, Wave Field Synthesis, Circular Harmonics
I-INCE Classification of Subjects Number(s): 38.2

1. INTRODUCTION
The bottom of figure 1 shows a typical one-dimensional ANC setup in a pipe. The top of the figure depicts an analog setup for a transmission line. In case of the transmission line a voltage pulse travelling from left to right would cause a voltage drop over the impedance $Z$ attached to the end of the line. Instead, a source introduces the same amount of current that would run through the impedance $Z$ caused by the voltage pulse, but with negative sign. Hence, there is zero voltage drop over the attached impedance $Z$ and twice the current $I$ at the end of the line. This also means that there is a termination with a reflectance factor of -1 as if the line was short-circuited.

In case of the pipe model, a one-dimensional primary field is propagating from left to right. A reference sensor measures this field and an ANC system should then minimize the pressure with the secondary speaker. The speaker itself behaves like the current source in the picture at the top of Figure 1. By introducing the correct amount of velocity $v$, the sound pressure will be zero. Left of the secondary source the resulting velocity then is two times as high as without the ANC system. Therefore, secondary sources of a perfectly working ANC system behave like a sound soft reflector or scatterer.

2. Reflection Equivalence in a Circular ANC Setup

The principle of the reflection equivalence can also be applied to a circular two-dimensional ANC system. To determine the appropriate secondary source functions, the velocity on a circular boundary condition has to be identified for a given two-dimensional primary field, since the source strengths are proportional to that velocity. Due to the circular symmetry, the following derivations are in polar coordinates.

2.1 Circular System Setup

Figure 2 shows a typical circular ANC setup on the left side. The outer circle consists of cardioid microphones pointing outward. The inner circle is a loudspeaker array to synthesize the secondary
field. The microphone circle has the radius $R_M$ and the circular speaker array has the radius $R_L$. The right side of Figure 2 depicts the same setup with a circular boundary condition having the contour $L$ instead of the secondary sources. The primary sound field is recorded by the microphone array. Therefore, the next step is to predict the velocity on the contour $L$ from the microphone signals.

\[
\mathcal{M}_\nu(\omega) = \frac{1}{2} \left( \frac{P_\nu(R_M, \omega) + \rho c V_{n,\nu}(R_M, \omega)}{H^{(1)}_\nu(kR_M) + H^{(2)}_\nu(kR_M) - j H^{(1)}_\nu(kR_M) - j H^{(2)}_\nu(kR_M)} \right)
\]  

(1)

In equation 1 the Greek letter $\nu$ is the order of the coefficient, $\omega$ is the angular frequency, $\rho$ is the density of air, $c$ the speed of sound in air and $k$ is the wavenumber. The denominator of equation 1 consists of Hankel’s functions of the first and second kind and their derivatives. The signal of a cardioid microphone is assumed to be the sum of the pressure $P_v$ and the normal velocity $V_{n,\nu}$. In practice, this sum does not need to be calculated since the cardioid microphones will supply the signal directly. Please note that directional patterns of high orders can be implemented with a circular microphone array. Hence, for a given order $\nu$ the values of $P_v$ and $V_{n,\nu}$ are calculated by weighting every single microphone signal with the corresponding term of the directional pattern and then building the sum of all products (1).

After computing the circular extrapolation coefficients for all frequencies $\omega$, it is possible to extrapolate the given primary field to any point $P(r, \varphi)$ in the two-dimensional space using (1)

\[
P(r, \varphi, \omega) = \sum_{\nu=-\infty}^{\infty} \mathcal{M}_\nu(\omega) \left[ H^{(1)}_\nu(kr) + H^{(2)}_\nu(kr) \right] e^{j\nu\varphi}
\]  

(2)
The next step is to introduce the circular boundary condition. In Figure 2 it is depicted by the grey circle with the radius \( R_L \) on the left side of the figure. Equation 3 is a general description of the sound field and \( A_\nu \) and \( B_\nu \) are yet unknown circular harmonics functions that have to be determined.

\[
P(r, \varphi, \omega) = \sum_{\nu=-\infty}^{\infty} \left[ A_\nu(\omega)H_\nu^{(1)}(kr) + B_\nu(\omega)H_\nu^{(2)}(kr) \right] e^{j\nu\varphi}, \tag{3}
\]

Generally, \( A_\nu(\omega)H_\nu^{(1)}(kr) \) represents converging and \( B_\nu(\omega)H_\nu^{(2)}(kr) \) diverging parts of the field. Assuming an impedance \( Z_L \) for the circular boundary condition, the following relation is valid on its contour \( L \) (see also (2)):

\[
P(R_L, \varphi, \omega) = \frac{-jZ_L}{\rho c} \left. \frac{\partial P(r, \varphi, \omega)}{\partial r} \right|_{r=R_L}. \tag{4}
\]

Applying equation (4) on equation (3) yields

\[
B_\nu(\omega) = -A_\nu(\omega) \cdot \frac{H_\nu^{(1)}(kR_L) + j\frac{Z_L}{\rho c} H_\nu^{(2)}(kR_L)}{H_\nu^{(2)}(kR_L) + j\frac{Z_L}{\rho c} H_\nu^{(2)}(kR_L)}. \tag{5}
\]

Furthermore, the function \( R_\nu \) is defined as the circular reflection coefficient:

\[
R_\nu(\omega) = \frac{H_\nu^{(1)}(kR_L) + j\frac{Z_L}{\rho c} H_\nu^{(2)}(kR_L)}{H_\nu^{(2)}(kR_L) + j\frac{Z_L}{\rho c} H_\nu^{(2)}(kR_L)}. \tag{6}
\]

After the substitution the right side of equation (6) in equation (5) by \( R_\nu \), equation (3) becomes

\[
P(r, \varphi, \omega) = \sum_{\nu=-\infty}^{\infty} A_\nu(\omega) \left[ H_\nu^{(1)}(kr) - R_\nu(\omega)H_\nu^{(2)}(kr) \right] e^{j\nu\varphi}. \tag{7}
\]

The function \( B_\nu \) has now been eliminated from the equation and all diverging parts of the field can be calculated according to the impedance \( Z_L \) and the function \( A_\nu \) of the converging parts. But \( A_\nu \) is still unknown. To determine \( A_\nu \), it is possible to investigate the case when the circular boundary condition (i.e. the circle) becomes infinitesimal. The circular reflection coefficient then has the value.
Please refer to (3, Appendix B1) for a detailed derivation of this result. With \( R_v(\omega) = -1 \) it is possible to rewrite equation 7 for the case that no boundary condition is present:

\[
P(r, \varphi, \omega) = \sum_{\nu=-\infty}^{\infty} A_\nu(\omega) \left[ H_\nu^{(1)}(kr) + \frac{jZ_L}{pc} H_\nu^{(2)}(kr) \right] e^{j\nu \varphi}.
\]  

Comparing equation 9 with equation 2 immediately shows that the function \( A_\nu \) for the converging parts of the field has to be equal to the circular expansion coefficient \( \mathcal{M}_\nu \):

\[
A_\nu(\omega) = \mathcal{M}_\nu(\omega).
\]  

Finally, it is possible to describe the complete two-dimensional field with a circular boundary condition of radius \( R_L \) and with the impedance \( Z_L \) present:

\[
P(r, \varphi, \omega) = \sum_{\nu=-\infty}^{\infty} \mathcal{M}_\nu(\omega) \left[ H_\nu^{(1)}(kr) - R_v(\omega) H_\nu^{(2)}(kr) \right] e^{j\nu \varphi}.
\]  

Applying Euler’s equation to equation 11 it is also possible to find a formulation for the calculation of the normal sound velocity in any point of the field with the circular boundary condition present:

\[
V_n(r, \varphi, \omega) = \frac{j}{\rho \omega} \sum_{\nu=-\infty}^{\infty} \mathcal{M}_\nu(\omega) \left[ H_\nu^{(1)}(kr) - R_v(\omega) H_\nu^{(2)}(kr) \right] e^{j\nu \varphi}.
\]  

### 2.4 Synthesis of Secondary Speaker Signals

First, it is necessary to recall the simple source formulation, which is described in (4). Assuming a continuous source distribution on the contour \( L \) of the grey circle in figure 2, the following equation can be used to determine the pressure in an arbitrary point at \( r \):

\[
P(r) = \int_L a(r_L) G(r|r_L) dL, \quad r \in \mathbb{R}^2.
\]  

Where \( G \) is the free field Green’s function and the source function \( a(r_L) \) is determined by the normal derivations of the exterior field \( P_e \) and the interior field \( P_i \) of the contour \( L \) on \( L \):

\[
a(r_L) = \nabla_n P_e(r_L) - \nabla_n P_i(r_L).
\]  

This property was used in (5) to derive the secondary source signals for wave field synthesis. It was also presumed that a Dirichlet boundary condition exists on \( L \), which is the case for a sound soft reflector. The total field on \( L \) then is:

\[
P_{\text{total}}(r_L) = P_{\text{primary}}(r_L) + P_{\text{reflected}}(r_L) = 0.
\]  

Now the gradient of the total field \( P_{\text{total}} \) on \( L \) is the sought-after source function \( a(r_L) \):

\[
a(r_L) = \nabla_n P'_{\text{primary}}(r_L) + \nabla_n P'_{\text{reflected}}(r_L) = \nabla_n P'_{\text{total}}(r_L).
\]  

The free field Green’s function in two dimensions is given by

\[
G_{2D}(r|r_L) = \frac{j}{4} H_0^{(2)}(k|r-r_L|).
\]  

Additionally, with Euler’s equation the normal sound velocity \( V_n_{\text{total}} \) of the total field can be used instead of the pressure gradient \( P_{\text{total}} \). Equation (12) then can be written as

\[
P(r, \omega) = \frac{j}{4} \int_L -j\omega \rho V_n_{\text{total}}(r_L, \omega) H_0^{(2)}(k|r-r_L|) dL.
\]  

Hence, to synthesize the secondary field it is necessary to calculate the normal velocity on the sound soft reflector. This can be achieved with equation (12), using the circular reflection coefficient \( R_v \) for a sound soft reflection \( (Z_L = 0) \) on the contour \( L \):

\[
R_v(\omega) = \frac{H_\nu^{(1)}(kR_L)}{H_\nu^{(2)}(kR_L)}.
\]
3. Simulation Results

Figure 3 presents the simulation results of the algorithm described in chapter 2. The simulated ANC system consists of 24 cardioid microphones (red crosses) and 24 speakers (red dots). The system’s reactions to three different types of primary fields at 300 Hz were investigated: a plane wave, an in z-direction infinite line source outside the ANC system and an in z-direction infinite line source inside the ANC system. The plane wave propagates from left to right under an angle of 30° with respect to the x-axis. Both the plane wave and the field of the line source outside the system are reproduced very exactly inside by the secondary sources with inverted phase. Therefore, the superpositions of the primary and secondary fields lead to destructive interference inside the loudspeaker circle, producing a quiet zone. Figure 4 shows the attenuation levels for the three situations. As can be seen, the dampening of the plane wave and the outside source are way over 60 dB. The two figures also show that the algorithm is immune to sound sources lying inside the quiet zone. Only slight reactions are visible.

4. Conclusions

The simulation results in Figure 3 and Figure 4 prove the ability of an ANC system to attenuate a given primary field significantly by incorporating transfer functions that where derived from the reflection equivalence. In comparison to a former implementation of the system based on the Kirchhoff-Helmholtz integral (see for example 6), the reflection equivalence does not depend on microphone pairs to measure the pressure and its gradient or respectively the velocity. It is sufficient to use cardioid microphones. With regard to the implementation on a digital signal processor (DSP), this directly reduces the mandatory number of input channels and processing power.

It is also possible to implement different system geometries with the reflection equivalence. Since analytic solutions for more complicated geometries than circles or squares are usually hard to find, simulation methods like FDTD or FEM can be utilized to “measure” the transfer functions inside a computer simulation (3, 5).

REFERENCES

Figure 3 Simulation of a two-dimensional ANC system using the reflection equivalence at 300 Hz. The red crosses depict the reference microphones and the red dots represent the secondary speaker positions. Lines (a) and (b) show the primary and the secondary field, line (c) shows the superposition of both. The left column illustrates the system’s reaction to a plane wave propagating from left to right under an angle of 30°. The primary field in the middle column was produced by an in z-direction infinite line source at $x = -1.7$ m and $y = -1$ m. The right column shows that the ANC system (or the algorithm) does not react on a line source lying inside of it.
Figure 4 Attenuation results of the simulations for the three different primary fields. The red crosses represent the reference sensors and the red dots the secondary sources of the ANC system.