Parameter Identification and Vibration Analysis of a Three-Dimensional Elastic Ring–Based Tire Model

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ABSTRACT
We present a new approach for predicting tire tread vibrations at frequencies less than 200 Hz; this approach is based on a three-dimensional elastic ring model, where the ring represents the tread and the springs represent the tire sidewall stiffness. The equations of motion for lateral, longitudinal, and radial vibrations on the tread are derived assuming inextensional deformation. Several associated parameters were identified experimentally through impact testing. Unlike most ring models, which only consider radial–circumferential modes, the presented model can also predict lateral bending modes. The experimental results agreed well with the analytical predictions.

1. INTRODUCTION
Road noise is one of the criteria affecting noise, vibration, and harshness (NVH) performance [1]. Car tires strongly influence the NVH experienced by the passengers; in other words, tire vibration characteristics influence the road noise spectra [2]. Recent studies on NVH prediction have presented tire finite element modeling that considers not only radial modes but also lateral vibration modes, such as lateral translational and lateral bending modes, all of which affect interior noise [3-6]. Above models are computationally expensive. The complex physically based models such as FE models are better suited to examine the effects of the change of a physical parameter such as material stiffness and details of tire structure[7]. On the other hand, computationally cheap models such as flexible ring models are suitable for use in general vehicle dynamics simulations. However, flexible ring models cannot predict lateral vibrations [8-11].

This paper presents a new approach for tire vibration analysis that considers lateral vibration and reports a formula for steady response of tires. First, we developed a three-dimensional low–degree of freedom dynamic model on the basis of the thin cylindrical shell theory. Kirchhoff–Love hypothesis is assumed to be valid for the proposed model. The basic equations, including the equation for the effect of initial tension caused by inflation pressure, were derived using the aforementioned theory and the Lagrange equation. Second, we derived the equation for the steady response of the tread center and conducted impact testing. Several parameters of the proposed model were identified by comparing the experimental and analytical frequency response functions.

2. Vibration analysis using the three-dimensional flexible ring model

2.1 Three-dimensional flexible ring model
The following tire modeling assumes a non-loaded and non-rolling condition. Figure 1 illustrates the tire dynamic model that is based on thin cylindrical shell theory. The proposed model is a three-dimensional flexible ring model in which a thin cylindrical shell ring and springs represent the tire tread and sidewall, respectively. The lateral, circumferential, and radial stiffness of the sidewall are distributed through the springs, which connect the edges of the tire tread to the edges of the wheel. The wheel is a rigid body owing to its high stiffness relative to the tire. Locations of tread-ring

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elements are described using a cylindrical coordinate system \((y, \theta, z)\), in which \(z = 0\) is considered the neutral plane of the tread ring and the vibrational displacement of an arbitrary point on the neutral plane with respect to \((y, \theta, z)\) is \((u, v, w)\). The parameters of the presented model are as follows: tread ring—radius \(R\), thickness \(b\), width \(2l\), modulus of longitudinal elasticity \(E\), tension \(S_0\), second moment of area \(I\), Poisson ratio \(\nu\), and mass density \(\rho\); sidewall—radial, circumferential, and lateral stiffness \(K_r, K_\theta, K_y\), respectively. Structural dampers are expressed as [12]

\[
S_0^* = (1 + j \eta_0)S_0
\]

\[
K_r^* = (1 + j \eta_r)K_r
\]

\[
K_\theta^* = (1 + j \eta_\theta)K_\theta
\]

\[
K_y^* = (1 + j \eta_y)K_y
\]

Given the thinness of the tread compared with the full tread radius, the lateral strain of the tread ring \(\varepsilon_y\), circumferential strain of the tread ring \(\varepsilon_\theta\), and shear strain of the neutral plane \(\gamma_{y\theta}\) are assumed to be 0. In addition, inextensional tread deformation is assumed [13]. The aforementioned strains are given by

\[
\varepsilon_y = \frac{\partial u}{\partial y} = 0
\]

\[
\varepsilon_\theta = \frac{\partial v}{R \partial \theta} + \frac{w}{R} = 0
\]

\[
\gamma_{y\theta} = \frac{\partial u}{R \partial \theta} + \frac{\partial v}{\partial y} = 0
\]

2.2 Derivation of the steady response

2.2.1 Mechanical energy

The total kinetic energy of the tread ring \(T\), which includes the lateral, circumferential, and radial velocity components, are given by

\[
T = \sum_{i=0}^{2\pi} \int_0^1 \int_2 \rho b \left( \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial v}{\partial t} \right)^2 + \left( \frac{\partial w}{\partial t} \right)^2 \right) Rd\theta dy dt
\]

The potential energy of tread ring bending \(U_1\) is [13],

\[
U_1 = \sum_{i=0}^{2\pi} \int_0^1 \int_2 \frac{EI}{2R^4} \left( \frac{\partial^2 w}{\partial \theta^2} + w \right)^2 Rd\theta dy dt
\]

The potential energy of tread ring torsion \(U_2\) is [13],
\[ U_2 = \int_0^{2\pi l} \frac{EI(1-\nu)}{R^2} \left( \frac{\partial^2 w}{\partial \theta \partial \theta} + \frac{\partial v}{\partial y} \right) R \, dy \, d\theta \]  

(10)

Internal pressure \( S_0 \) induces tension in the tread ring. The potential energy of tread ring tension \( U_3 \) is \([11]\).

\[ U_3 = \int_0^{2\pi l} -\frac{S_0^*}{2R^2} \left( \frac{\partial^2 w}{\partial \theta^2} + w \right) R \, dy \, d\theta \]  

(11)

The potential energy stored in the lateral, circumferential, and radial springs, respectively, is

\[ U_4 = \int_0^{2\pi l} \frac{1}{2} K\theta u_{y=0}^2 R \, d\theta + \frac{1}{2} \int_0^{2\pi l} K\theta u_{y=\theta}^2 R \, d\theta \]  

(12)

\[ U_5 = \int_0^{2\pi l} \frac{1}{2} K\theta v_{y=0}^2 R \, d\theta + \frac{1}{2} \int_0^{2\pi l} K\theta v_{y=\theta}^2 R \, d\theta \]  

(13)

\[ U_6 = \int_0^{2\pi l} \frac{1}{2} K\theta w_{y=0}^2 R \, d\theta + \frac{1}{2} \int_0^{2\pi l} K\theta w_{y=\theta}^2 R \, d\theta \]  

(14)

When an external force \( F_u \) acts in the horizontal direction, the induced energy \( W_u \) is given by Equation (15), where \( u \) is the horizontal displacement. Similarly, when external forces \( F_v \) and \( F_w \) act in the circumferential and radial directions, respectively, the induced energy \( W_v \) and \( W_w \) are given by the Equations (16) and (17), respectively, where \( v \) and \( w \) are displacements in the circumferential and radial directions, respectively.

\[ W_u = \int_0^{2\pi l} \int_0^{2\pi R} F_u R \, dy \, d\theta \]  

(15)

\[ W_v = \int_0^{2\pi l} \int_0^{2\pi R} F_v R \, dy \, d\theta \]  

(16)

\[ W_w = \int_0^{2\pi l} \int_0^{2\pi R} F_w R \, dy \, d\theta \]  

(17)

External forces \( F_u, F_v, \) and \( F_w \) in the transverse, circumferential, and radial directions, respectively, are given by the following equations, where \( \theta = \theta_0, y = y_0 \).

\[ F_u = -F \delta(\theta - \theta_0) \delta(y - y_0) \cos \omega t \]  

(18)

\[ F_v = -F \delta(\theta - \theta_0) \delta(y - y_0) \cos \omega t \]  

(19)

\[ F_w = -F \delta(\theta - \theta_0) \delta(y - y_0) \cos \omega t \]  

(20)
2.2.2 Equations of motion and steady response

Steady responses of lateral translation, lateral bending, in-plane torsion, and radial mode are derived as explained in this section. First, displacement in each mode is defined using the general coordinates \( \alpha(t) \), which are expressed using amplitude \( a \) and angular velocity \( \omega \). Second, the equation of motion is derived using Lagrange’s method, where the Lagrangian \( L \) is determined by substituting the defined displacement into the energy equation. Finally, compliance is demanded by calculating from that steady response of displacement divided by external force, and the steady response is obtained by solving the equation of motion for amplitude.

The displacement, Lagrangian, and steady response of lateral translation are obtained using the following equations:

\[
\begin{align*}
\begin{cases}
  u = -a(t) \\
  v = 0 \\
  w = 0
\end{cases}
\end{align*}
\]  

\[ L = 2\pi R \rho b l \alpha(t)^2 - 2\pi R K_y^* \alpha(t)^2 + F \cos \omega t \cdot \alpha(t) \]  

\[ u(\theta, y) = \frac{-F}{(K_{nu} - \omega^2 M_{nu})} \cos \omega t \]  

\[ M_{nu} = 4\pi R \rho b l \]  

\[ K_{nu} = 4\pi R K_y^* \]  

Similarly, the corresponding equations for lateral bending are

\[
\begin{align*}
\begin{cases}
  u = -\frac{R}{n} \{ \alpha_n(t) \cos n\theta - \beta_n(t) \sin n\theta \} \\
  v = y \{ \alpha_n(t) \sin n\theta + \beta_n(t) \cos n\theta \} \\
  w = -ny \{ \alpha_n(t) \cos n\theta - \beta_n(t) \sin n\theta \}
\end{cases}
\end{align*}
\]  

\[ L = \pi \rho b l \left( \frac{R^2}{n^2} + \frac{l^2}{3} + \frac{l^2 n^2}{3} \right) \{ \alpha_n^2(t) + \beta_n^2(t) \} \]  

\[ - \pi R \left[ \frac{S_0}{3R^2} l^2 n^2 (n^2 - 1) + \frac{R^2}{n^2} K_y^* + l^2 K_\theta^* + n^2 l^2 K_r^* \right] \{ \alpha_n^2(t) + \beta_n^2(t) \} \]  

\[ + \frac{R}{n} F \cos \omega t \{ \alpha_n(t) \cos n\theta_0 - \beta_n(t) \sin n\theta_0 \} \]  

\[ u(\theta, y) = \frac{-R^2 F}{n^2 (K_{bn} - \omega^2 M_{bn})} \cos n(\theta - \theta_0) \cos \omega t \]  

\[ M_{bn} = 2\pi \rho b l \left( \frac{R^2}{n^2} + \frac{l^2}{3} + \frac{l^2 n^2}{3} \right) \]
The corresponding equations for in-plane torsion are

\[
\begin{align*}
  u &= 0 \\
  v &= -\alpha(t) \\
  w &= 0
\end{align*}
\]  

(31)

\[
L = 2\pi\rho bl\alpha(t)^2 - 2\pi R\kappa_\alpha(t)^2 + F \cos \omega t \cdot \alpha(t)
\]  

(32)

\[
v(\theta, y) = \frac{-F}{(K_{\text{tor}} - \omega^2 M_{\text{tor}})} \cos \omega t
\]  

(33)

\[
M_{\text{tor}} = 4\pi\rho bl
\]  

(34)

\[
K_{\text{tor}} = 4\pi\kappa_\alpha
\]  

(35)

Finally, the corresponding equations for radial mode are

\[
\begin{align*}
  u &= 0 \\
  v &= \{\alpha_n(t) \sin n\theta + \beta_n(t) \cos n\theta\} \\
  w &= -n[\alpha_n(t) \cos n\theta - \beta_n(t) \sin n\theta]
\end{align*}
\]  

(36)

\[
L = \pi\rho bl(n^2 + 1)\{\dot{\alpha}_n^2(t) + \dot{\beta}_n^2(t)\}
\]  

\[ - \pi R \frac{S^*}{R^2} l n^2 (n^2 - 1) + K_\alpha^* + n^2 K_r^* \{\dot{\alpha}_n^2(t) + \dot{\beta}_n^2(t)\}
\]  

\[ + F \cos \omega t \{\alpha_n(t) \cos n\theta - \beta_n(t) \sin n\theta\}
\]  

(37)

\[
v(\theta, y) = \frac{-F}{(K_{r,n} - \omega^2 M_{r,n})} \cos(n(\theta - \theta_0)) \cos \omega t
\]  

(38)

\[
M_{r,n} = 2\pi\rho bl(n^2 + 1)
\]  

(39)

\[
K_{r,n} = 2\pi R \frac{S^*}{R^2} l n^2 (n^2 - 1) + K_\alpha^* + n^2 K_r^*
\]  

(40)

Thus, the steady response formula of displacement along the lateral, circumferential, and radial directions can be obtained by superposing the aforementioned derived equations, as follows:
\[ u(\theta, y) = \frac{-F}{(K_{ra} - \omega^2 M_{ra})} \cos \omega t + \frac{-R^2 F}{n^2(K_{b,n} - \omega^2 M_{b,n})} \cosn(\theta - \theta_n) \cos \omega t \] \hspace{1cm} (41)

\[ v(\theta, y) = \frac{-F}{(K_{or} - \omega^2 M_{or})} \cos \omega t + \frac{-F}{(K_{r,n} - \omega^2 M_{r,n})} \cosn(\theta - \theta_n) \cos \omega t + \frac{-y_0 F}{(K_{b,n} - \omega^2 M_{b,n})} \cos(\theta - \theta_n) \cos \omega t \] \hspace{1cm} (42)

\[ w(\theta, y) = \frac{-n^2 F}{(K_{r,n} - \omega^2 M_{r,n})} \cosn(\theta - \theta_n) \cos \omega t + \frac{-n^2 F}{(K_{b,n} - \omega^2 M_{b,n})} \cosn(\theta - \theta_n) \cos \omega t \] \hspace{1cm} (43)

### 2.3 Derivation of the frequency response function

Assuming the experiment of three vibration modes, three accelerations are shown here. The experimentally obtained frequency response functions are accelerances. Additional data on the input and responses are presented in Chapter 3. Let \( F \) and \( a \) represent the magnitude of the force imposed by the impact hammer and the acceleration, respectively. In all cases, the response point is the center of the tread.

Accelerance in the lateral, circumferential, and radial modes are given by Equations (44)–(46), respectively.

\[ \frac{a_u(\theta_0, 0)}{-F \cos \omega t} = \frac{\omega^2}{K_{ra} - \omega^2 M_{ra}} + \frac{R^2 \omega^2}{n^2(K_{b,n} - \omega^2 M_{b,n})} \] \hspace{1cm} (44)

\[ \frac{a_v(\theta_0, 0)}{-F \cos \omega t} = \frac{\omega^2}{K_{or} - \omega^2 M_{or}} + \frac{\omega^2}{K_{r,n} - \omega^2 M_{r,n}} \] \hspace{1cm} (45)

\[ \frac{a_w(\theta_0, 0)}{-F \cos \omega t} = \frac{n^2 \omega^2}{K_{r,n} - \omega^2 M_{r,n}} \] \hspace{1cm} (46)

### 3. Vibration test

Frequency response function was measured experimentally through hammering tests. Figure 2 presents the schematic of hammering test. A non-loaded and non-rolling 195/65R15 tire fixed to an axle and inflated to the standard pressure of 220 [kPa] was used. Figure 3 presents the excitation point and direction as well as the response point. In the lateral translation and lateral bending modes, the excitation point was at the end of the line of the accelerometer; the impact was in the lateral direction. In the in-plane torsion mode, the excitation point was on a tread-mounted small block on the tread center immediately beside the accelerometer; the impact direction was circumferential. In the radial mode, the excitation point was immediately beside the accelerometer and on the tread center; the impact direction was radial. In all cases, the input was given using an impact hammer (PCB 086C03), and the response was measured using an accelerometer (PCB 372A73). The measured data were analyzed using a frequency analyzer (Ono Sokki DataStation2000). The experiment was repeated five times in each mode, and the frequency response function was calculated as the average of the five measurements. The measurement frequency range was up to 400 [Hz], and the frequency resolution was 0.125 [Hz].
4. Parameter identification and verification

As stated earlier, the parameters of the proposed three-dimensional elastic ring model are $S_0$, $K_y$, $K_\theta$, $K_r$, $\eta_y$, $\eta_\theta$, $\eta_r$, and $\rho b$. Note that $l$ and $R$ are approximate for a given tire shape, and $\nu$ is a constant equaling 0.3. These initial values were input in the numerical analysis to calculate the analytical frequency response functions (hereafter, $FRF_{ana}$) using Equations (44)–(46). Subsequently, $FRF_{ana}$ and the experimentally obtained frequency response functions (hereafter, $FRF_{exp}$) were input in Equation (47), and parameter identification was performed until the combination of parameters that yields the minimum $J$ was identified.

$$J = \sum \frac{FRF_{exp}^4}{f^2} \left( \frac{FRF_{exp} - FRF_{ana}}{FRF_{exp}} \right)^2$$  \hspace{1cm} (47)

Figure 4(a)–(c) presents the analytical and experimental FRFs in the lateral, torsion, and radial modes, respectively. $FRF_{ana}$ in all three modes agreed well with the corresponding $FRF_{exp}$. Table 1 lists the identified model parameters.
(a) Lateral translation and bending mode

(b) In-plane torsion mode

(c) Radial mode

Figure 3 – Analytical and experimental FRFs

Table 1 – Identified model parameters

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<th>Tire parameter</th>
<th>Identified value</th>
<th>Unit</th>
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<td>$K_r$</td>
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<td>[N/m$^2$]</td>
</tr>
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<td>$K_\theta$</td>
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<td>[N/m$^2$]</td>
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5. CONCLUSIONS

This paper reported a new approach for predicting tire tread vibrations that uses a three-dimensional elastic ring–based model. The basic equations, including that for the effect of initial tension caused by inflation pressure, were derived on the basis of thin cylindrical shell theory and the Lagrange equation. Next, we derived the steady response equations of the tread center and conducted an impact test in the lateral translation, lateral bending, in-plane torsion, and radial modes. Parameter identification for the proposed model was performed by comparing the frequency response functions obtained analytically and experimentally. Therefore, this model can be used for comprehensive vibrational analysis of tires.

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