A Multi-Model Reduction Technique for Optimization of Coupled Structural-Acoustic Problems

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ABSTRACT
Finite Element models of structural-acoustic coupled systems can become very large for complex structures with multiple connected parts. Optimization of the performance of the structure based on harmonic analysis of the system requires solving the coupled problem iteratively and for several frequencies, which can become highly time consuming. Several modal-based model reduction techniques for structure-acoustic interaction problems have been developed in the literature. The unsymmetric nature of the pressure-displacement formulation of the problem poses the question of how the reduction modal base should be formed, given that the modal vectors are not orthogonal due to the asymmetry of the system matrices. In this paper, a multi-model reduction (MMR) technique for structure-acoustic interaction problems is developed. In MMR, the reduction base is formed with the modal vectors of a family of models that sample the design domain of the optimization parameters. The orthogonalization of the resulting reduction base is therefore a key point in the method. The use of the different reduction approaches found in the literature for developing an efficient and robust MMR technique is investigated. Several methods are compared in terms of accuracy and size of the reduced systems for optimization of simple models.

Keywords: Model Reduction, Optimization, Structure-acoustic interaction

1. INTRODUCTION

Significant computational challenges are encountered when solving numerical problems that require iterative and/or repeated calculations of large complex models. In our investigations, we focus on the challenges encountered in the field of hearing aids, which are devices composed of a large number of small parts with complex dynamic and acoustical behavior. Numerical vibro-acoustic analysis of hearing aids is essential for the study of problems such as feedback, which is currently the main gain limiting factor of the hearing devices, and requires fine resolution frequency calculations. Tasks such as uncertainty analysis by means of Monte-Carlo methods, or parametric and topology optimization, require solving the system repeatedly for a large number of variations of different parameters. Therefore, the time required for solving the numerical problem at each iteration becomes a critical factor.

Recently, a topology optimization study including structure-acoustic interaction was performed on a part of a hearing instrument [1]. To facilitate the study, some restrictions on the design freedom were imposed and the performance was evaluated and optimized for a limited number of frequencies. The effects of such simplifications are difficult to control; therefore, there is a need to develop computational reduction techniques that make performing these processes with the required level of accuracy practically possible.

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Reduction of Finite Element (FE) models of hearing aids is quite an unexplored field. The main challenges of modelling sound and vibrations in hearing aids have been discussed in Ref. [2], where a complete model of a hearing aid was developed for the purpose of studying the feedback paths. A pragmatic 3-D model was suggested, which uses the FE method combined with other methods including structural fuzzy [3] (describing regions with large uncertainty), lumped elements, two-port acoustical networks and measured data. These methods describe some of the parts of the hearing aid in a more efficient way than a pure FE model. However, when a high accuracy is required, the details of complex geometries need to be included in the model, which requires pure FE modeling that results in large matrices.

Model reduction techniques have been described in the literature and applied in several fields. The Component Mode Synthesis (CMS) methods have been widely used for reducing large complex structural problems. They are based on decomposing big structures into several substructures, which are then described by a number of modes, and linked together by shared degrees of freedom [4, 5]. The method has also been extended to describe problems with structure-acoustic interaction by several authors, who have taken different approaches when constructing the reduction base. One approach is using the uncoupled structural and acoustic modes, which, for problems with weak coupling, can be efficiently applied. However, for problems with strong interaction, a large amount of modes would be required in order to obtain an accurate reduced model, as described in Ref. [6]. Methods that use coupled modes for the reduction have also been developed. A drawback of those methods is that solving the coupled eigenvalue problem requires higher computational effort due to the unsymmetric nature of the matrices when the pressure-displacement formulation is used. However, the coupled modal vectors can better describe the system, and smaller reduction bases can be obtained. The asymmetry of the matrices also implies that the modal vectors do not form an orthogonal base, which is a basic requirement for forming a vector base. This issue can be solved by using both left and right eigenvectors [7], or by applying orthogonalization techniques [8].

In CMS methods, the interface degrees of freedom between substructures need to be kept in the reduced set of coordinates. This makes them not suitable for structures where it is challenging to find small interfaces to split the model, such as fully coupled structural-acoustic models that include exterior acoustic domains. Even though interface reduction methods have been developed, such as in Refs. [9, 10], avoiding the interfaces would be desirable. In many cases, the available computational power is sufficient to allow for solving the full model eigenvalue problem, which provides the global modal vectors. Those can be used as a reduction base for the system matrices, allowing for the calculation of fine resolution frequency responses at a reduced computational cost. However, for optimization purposes, the reduction base should be re-calculated at each iteration (unless the sensitivity of the modal vectors to the optimization variables is low enough, as assumed in Refs. [11, 12]), which would still be highly time-consuming.

The Multi-Model Reduction (MMR) technique consists in constructing a reduction base with modal vectors from a family of models, formed by sampling the design domain of the optimization parameters. The sampling should be fine enough so that the design points that are not sampled are approximated accurately enough, therefore the efficiency of this method depends on how many different models need to be included in the reduction base. This technique has been described for structural problems in Ref. [13], where the need of orthogonalizing the reduction base is highlighted. Since some modal vectors from different sampled models can be very similar, the linearly dependent vectors would originate ill-conditioning otherwise. When using MMR for coupled problems, the fact that the modal vectors resulting from the unsymmetric matrix systems are inherently not orthogonal needs to be taken into account on top of that.

This paper discusses how to adapt the MMR technique for coupled problems and reviews different approaches to form a numerically stable reduction base. In section 2, the different approaches are described, and in section 3, the efficiency of the different suggested methods is compared for the optimization of a simple model of a plate coupled to an air column.
2. METHODS

In this section, different approaches to reducing a coupled structure-acoustic interaction problem are described. For an undamped system, the pressure-displacement formulation yields the coupled eigenvalue problem

\[
\begin{bmatrix}
K_s & -[S]^T \\
0 & K_a
\end{bmatrix} - \omega_i^2 \begin{bmatrix}
[M_s] & [0] \\
[0] & [M_a]
\end{bmatrix} \begin{bmatrix}
\psi_{1R} \\
\psi_{1R}^T
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

(1)

where \([K_s]\) and \([M_s]\) are the structural stiffness and mass matrices, \([K_a]\) and \([M_a]\) are the acoustic stiffness and mass matrices, \([S]\) is the coupling matrix, \(\rho_a\) is the density of the acoustic medium, \(\omega_i\) is the \(i\)-th modal angular frequency, \(\{\psi_{1R}\}\) is the structural displacement part of the \(i\)-th right modal vector and \(\{\psi_{aR}\}\) is the acoustic pressure part of the \(i\)-th right modal vector.

For the sake of notation clarity in the following sections, let us introduce \([M]\) and \([K]\) as the coupled mass and stiffness matrices, \(\{\psi_{1R}\}\) as the \(i\)-th complete right eigenvector and \(\{\psi_{1L}\}\) as the \(i\)-th complete left eigenvector, the latter being the result from solving the eigenvalue problem

\[
([K]^T - \omega_i^2 [M]^T) \{\psi_{1L}\} = \{0\}
\]

(2)

It becomes clear that a disadvantage of using both left and right eigenvectors in the reduction is that two eigenvalue problems must be solved in order to construct the base. However, according to Refs. [14] and [7], the left eigenvectors can be calculated from the right eigenvectors as

\[
\{\psi_{1L}\} = (\{\psi_{1R}\}^T, \frac{1}{\omega_i^2} \{\psi_{aR}\}^T),
\]

(3)

which avoids the extra computational effort.

Moreover, let us define the truncated modal matrices as

\[
[\psi_{L}] = [[\psi_{1L}], ... \{\psi_{N_L}\}], [\psi_{R}] = [[\psi_{1R}], ... \{\psi_{N_R}\}], [\psi_{aL}] = [[\psi_{1aL}], ... \{\psi_{N_{aL}}\}], [\psi_{aR}] = [[\psi_{1aR}], ... \{\psi_{N_{aR}}\}] \text{ and } [\psi_{sR}] = [[\psi_{1sR}], ... \{\psi_{N_{sR}}\}],
\]

where \(N\) is the number of modes included in the modal matrix.

In section 2.1, different approaches to reduction bases for the coupled system are described, and in section 2.2 those methods are extended to form multi-model reduction bases.

2.1 Single Model Reduction

The right eigenvectors from unsymmetric eigenvalue problems are not orthogonal, which makes them not suitable as a reduction base. However, the left and right eigenvectors form an orthogonal base with respect to the unsymmetric mass matrix, \([\psi_L][M][\psi_R] = [I]\); therefore, they can be used together to reduce the system. Another option consists in orthogonalizing the set of right eigenvectors, as done in Ref. [8], where the acoustic and structural parts of the coupled eigenvectors are separated and orthogonalized with respect to the acoustic and structural mass matrices respectively. From these two basic approaches, 6 reduction methods have been designed in order to compare and determine the most efficient and accurate reduction technique for the cases at hand.
In the following, \([T_L]\) is the left reduction matrix, and \([T_R]\) is the right reduction matrix, meaning that the reduced system matrices are calculated as

\[
[M_{\text{red}}] = [T_L][M][T_R]
\]
\[
[K_{\text{red}}] = [T_L][K][T_R].
\]

(4)

(5)

The six methods are summarized in the following.

1. Method 1: Using only the right eigenvectors for both left and right reduction matrices,

\[
[T_L] = [\psi_R]
\]

\[
[T_R] = [\psi_R].
\]

(6)

(7)

2. Method 2: Using only the left eigenvectors for both left and right reduction matrices,

\[
[T_L] = [\psi_L]
\]

\[
[T_R] = [\psi_L].
\]

(8)

(9)

3. Method 3: Using the right eigenvectors for the right reduction matrix, and the left eigenvectors for the left reduction matrix,

\[
[T_L] = [\psi_L]
\]

\[
[T_R] = [\psi_R].
\]

(10)

(11)

4. Method 4: Using only right eigenvectors, separating the fields and orthogonalizing. The matrices are formed by orthogonalizing the acoustic and structural parts of the right eigenvectors with the acoustic and structural mass matrices respectively, so that

\[
[T_{aL}]^T[M_a][T_{aR}] = [I]
\]

\[
[T_{sL}]^T[M_s][T_{sR}] = [I],
\]

(12)

(13)

and including them separately in the reduction matrices

\[
[T_L] = \begin{bmatrix} [T_{sL}] & [0] \\ [0] & [T_{aL}] \end{bmatrix}
\]

\[
[T_R] = \begin{bmatrix} [T_{sR}] & [0] \\ [0] & [T_{aR}] \end{bmatrix}.
\]

(14)

(15)

In this way, the structure of the complete unsymmetric problem in eq. (1) is preserved in the reduced problem, but the base is not orthogonal with respect to the total mass or stiffness matrices.

Due to the separation of the two domains, some vectors can be very similar, and those must be removed from the base to avoid ill-conditioning. The detection of collinear vectors is done before the orthogonalization by performing a Singular Value Decomposition (SVD) of the matrices \([\psi_{aR}]^T[M_a][\psi_{aR}]\) and \([\psi_{sR}]^T[M_s][\psi_{sR}]\), and keeping those vectors for which the singular values are below a selected tolerance. A recommended value [15] is \(\text{tol} = \epsilon \sigma_1 n\), where \(\epsilon\) is the floating-point relative accuracy, \(\sigma_1\) is the biggest singular value and \(n\) is the length of the diagonal of the matrix. Since the SVD has already been calculated, the obtained matrices can be used for orthogonalizing the base. The process for orthogonalizing \([\psi_{aR}]\) is described in the following. The SVD is calculated, yielding

\[
[\psi_{aR}]^T[M_a][\psi_{aR}] = [U][\Sigma][V]^T,
\]

(16)
where $[\Sigma]$ is the matrix with the singular values on its diagonal, and $[U]$ and $[V]$ are unitary matrices. Then,

$$[T_{aR}] = [\psi_{aR}] [V]^{-T} [\Sigma]^{-1/2}$$

and

$$[T_{aL}] = [\Sigma]^{-1/2} [U]^{-1} [\psi_{aR}]^T$$

are orthogonal bases. Moreover, since $[U]$ and $[V]$ are also orthogonal matrices, $[U]^{-1} = [U]^T$ and $[V]^{-T} = [V]$, which avoids matrix inversions in the calculation. The same procedure is followed to obtain $[T_{sL}]$ and $[T_{sR}]$, but using $[\psi_{sR}]$ in eqs. (17-18).

5. Method 5: Using only left eigenvectors, separating the fields and orthogonalizing. The procedure is as in Method 4, but using the left eigenvectors, $[\psi_{aL}]$ and $[\psi_{sL}]$, in eqs. (17-18).

6. Method 6: Using both left and right eigenvectors, separating the fields and orthogonalizing. This method combines features from methods 3, 4 and 5 by forming two bases with right and left eigenvectors where the structural and acoustical parts of the vectors are separated,

$$[\hat{\psi}_L] = \begin{bmatrix} [\psi_{sL}] & [0] \\ [0] & [\psi_{aL}] \end{bmatrix}$$

and

$$[\hat{\psi}_R] = \begin{bmatrix} [\psi_{sR}] & [0] \\ [0] & [\psi_{aR}] \end{bmatrix},$$

and then orthogonalizing them with respect to the full coupled mass matrix, so that

$$[T_L][M][T_R] = [I].$$

The orthogonalization is done by following the same procedure as described for Method 4, but using $[\hat{\psi}_L]$ in eq. (17) and $[\hat{\psi}_R]$ in eq. (18). Therefore, in this case the reduction base is orthogonal with respect to the total mass matrix.

Since methods 1 and 2 use a non-orthogonal base, inaccurate results are expected due to ill-conditioning of the reduced matrices. To avoid that, in methods 4 and 5 an orthogonalization step is introduced, which is applied to the acoustic and structural parts independently. This should enhance the results, since the independent fields can present very similar shapes across vectors; however, the downside of the method is that the initial reduction base contains twice as many vectors due to the splitting of the fields, and even though the final size depends on the number of vectors that are eliminated in the orthogonalization procedure, the reduced matrices will be larger and therefore more time consuming to solve. Method 3 uses the left and right eigenvectors, which are orthogonal with respect to the mass matrix, and should therefore form a complete expansion base that reduces the system accurately without increasing the size of the problem. However, there might be a benefit from separating the acoustic and structural fields due to the large scaling differences between the pressure values of the acoustic field and the displacement values of the structural field; therefore, in Method 6, this concept is introduced on top of the Method 3 approach. The performance of the different methods is evaluated in section 3.1.

2.2 Multi-Model Reduction

In order to use these reduction methods in optimization procedures, the reduction base would have to be calculated at each iteration, since the modal vectors will change when the values of the optimization parameters vary. However, this would be highly time consuming, and it is desirable to pre-construct the reduction base.
outside the optimization loop. If the modal vectors are not highly sensitive to the optimization parameters, a reduction base could be formed with the modes of the initial model and used throughout the optimization; however, this is often not the case. Then, a reduction base can be formed with the modal vector matrices from several models where the design domain of the optimization parameters have been sampled.

The six methods described in section 2.1 should be modified by substituting the single-model modal matrices $[\psi_L]$ and $[\psi_R]$ by the multi-model modal matrices, formed as

$$
[\psi_L]_1 \ [\psi_L]_2 \ldots \ [\psi_L]_M \\
[\psi_R]_1 \ [\psi_R]_2 \ldots \ [\psi_R]_M ,
$$

(22)

$$
[\psi_L]_1 \ [\psi_L]_2 \ldots \ [\psi_L]_M \\
[\psi_R]_1 \ [\psi_R]_2 \ldots \ [\psi_R]_M ,
$$

(23)

where $[\psi_{L/R}]_j$ is the modal matrix of the $j$-th model and $M$ is the total number of models included in the base. Since some of the modal vectors from different models will be very similar, an orthogonalization step must be added to methods 1, 2 and 3 (since methods 4, 5 and 6 already included one). The orthogonalization for methods 1, 2 and 3 is done following the procedure described for Method 4, but orthogonalizing the multi-model modal matrices with respect to the total mass matrix.

The number of models that need to be included and the sampling criteria of the design domain will be model-dependent. In section 3.2, the procedure to form an efficient base for the optimization of a plate model is discussed.

3. RESULTS

A model of a plate coupled to an air column is considered for testing the methods. The dimensions of the plate are $L_x = 4$ cm, $L_y = 3$ cm, and the objective is to optimize its thickness, which can vary between $10^{-5}$ m and $10^{-2}$ m. The air column is $L_z = 5$ cm high, and its walls are rigid, except for the side that is coupled to the plate. The frequency range of interest is between 100 Hz and 10 kHz. $L_x$ is discretized with 20 elements, $L_y$ is discretized with 16 elements and $L_z$ is discretized with 25 elements, which means that the acoustic elements are hexahedra of $2 \times 1.9 \times 2$ mm. The plate elements are formulated with 3 DOFs per node (one displacement and two rotations), and the acoustic elements are formulated with 1 DOF per node, and using linear shape functions, with the total number of DOFs resulting in 10353.

In section 3.1, the 6 methods introduced in section 2.1 are tested for 18 plate models where the thickness of each of the elements in the plate is defined by a random number between the aforementioned limits. In section 3.2, the efficiency of the methods introduced in section 2.2 is evaluated for the same 18 models, and the number of models that are needed to form an accurate base for the optimization of the plate is determined. The selected reduction bases are then used for the optimization of the plate thickness, and the results are shown and compared to full-model optimization results in section 3.3.

The accuracy of the methods is evaluated in terms of the modal frequencies, the modal vectors and the pressure frequency response at a point of the acoustic column. The error between the modal frequencies from the full system and the reduced system is evaluated as

$$
f_{err} = \max_{n=1:N} \frac{|f_{n\text{full}} - f_{n\text{red}}|}{f_{n\text{full}}} \cdot 100, \quad (24)
$$

where $N$ is the highest modal frequency below the upper limit of 10 kHz. The modal vectors accuracy is evaluated by calculating the Modal Assurance Criterion (MAC) between the modal vectors resulting from the full and the reduced systems. The MAC matrix between two generic vector matrices $[\phi]_A$ and $[\phi]_B$ is calculated as

$$
[NAC]_{(i,j)} = \left| \frac{[\phi]^T_A [\phi]_B^2}{([\phi]^T_A [\phi]_A) ([\phi]^T_B [\phi]_B)} \right| \quad (25)
$$
for each vector pair \((i, j = 1: N)\). The MAC is bounded between 0 and 1, 1 indicating linearly dependent vectors, and 0 indicating orthogonal vectors. The minimum diagonal value of the MAC matrix of all modes below 10 kHz is used as accuracy indicator,

\[
v_{\text{acc}} = \min_{n=1:N} [\text{MAC}]_i(n, n).
\]  

(26)

The accuracy of the pressure frequency response at a point at the upper corner of the acoustic domain when the plate is excited by a perpendicular point force at its central point is evaluated as the mean Sound Pressure Level (SPL) error in dB between the results of the full system and the reduced system. When the frequency response is sampled logarithmically at \(K\) points between 100 Hz and 10 kHz, the error is calculated as

\[
\Delta L_p = \frac{1}{K} \sum_{k=1}^{K} 20 \log \left( \frac{|p_{\text{red}}(k)|}{|p_{\text{full}}(k)|} \right).
\]  

(27)

where \(k\) is the \(k\)-th frequency line, and \(K\) is the total number of frequencies included in the analysis.

### 3.1 Single Model Reduction

The accuracy of the six methods has been evaluated for 18 plate models, where the thickness of each plate element is defined by a random number between the given limits \((10^{-5} \text{ m} \text{ and } 10^{-2} \text{ m})\). The variation along 4 orders of magnitude is selected in order to test the effects of having a wide design space. The abrupt changes of thickness between neighbouring elements will have an effect on the acoustic domain which is not taken into account here, since this is not relevant for the purpose of testing the efficiency of the reduction methods. All modes under a given frequency limit are included in the reduction base. In this case, the limit has been varied between 10 kHz and 20 kHz, but no benefit of extending the frequency range above the upper considered frequency (10 kHz) can be seen in the results. Figure 1 shows the spread of the accuracy of each method for the 18 models, in terms of modal frequencies, modal vectors and SPL, calculated as in eqs. (24-27), versus the computational time required for solving the reduced eigenvalue problem and calculating the frequency response at 300 frequency lines. As a reference, using the full system, that calculation takes about 2 min.

![Figure 1: Accuracy and time consumption of solving the single-model reduced system with the 6 methods for 18 different thickness conditions of the plate.](image)

The results show that Method 2 presents poor accuracy in terms of modal frequencies and vectors, and Method 1 shows the worst accuracy in terms of the frequency response. Method 3 is the least time consuming overall, and shows good accuracy in terms of modal frequencies and vectors, but presents significant SPL errors for some of the samples. Methods 4 and 5 present the best accuracy overall, however, they are more time consuming than the rest. Regarding Method 6, the accuracy is poor in terms of modal vectors for some
Looking at the calculated frequency responses for Method 3, it can be seen that the amplitude is correct around the peaks, and the errors arise around the antiresonances, which indicates that the errors may be due to a poor approximation of the phase shifts between modes. The choice of the best method for each application should be a compromise between required accuracy (where Methods 4 or 5 are best) and speed (where Method 3 is best).

3.2 Multi-Model Reduction

In order to construct a reduction base that can be used to optimize the plate, several models where the domain of the optimization parameters is sampled should be included. The thickness of each plate element is a parameter to be optimized, therefore there are $16 \times 20 = 320$ optimization parameters. Due to the large number of variables, a random sampling of the design space is used. Given that the parameter domain varies through 4 orders of magnitude ($10^{-5}$ mm to $10^{-2}$ mm), and the modal parameters are more sensitive to relative than linear changes of the design variables, the models for the reduction base are calculated as vectors of random numbers drawn from the uniform distributions

\[
\text{Model 1} : t \sim U([0, 10^{-2}]) \tag{28}
\]

\[
\text{Model 2} : t \sim U([0, 10^{-3}]) \tag{29}
\]

\[
\text{Model 3} : t \sim U([0, 10^{-4}]) \tag{30}
\]

\[
\text{Model 4} : t \sim U([0, 10^{-5}]). \tag{31}
\]

For higher number of models, the sequence is repeated (i.e. Model 5 is generated as Model 1, etc.). For each model, all modes below 10 kHz are included in the reduction base. In order to determine the number of models necessary for each of the 6 available methods to obtain an accurate reduced system, three criteria are considered,

\[
f_{err} < 1\% \tag{32}
\]

\[
v_{acc} > 0.9 \tag{33}
\]

\[
\Delta L_p < 1\text{dB} \tag{34}
\]

which are evaluated at the 18 different design points that were used in section 3.1. The results for the 6 methods have been analysed for a number of models ranging between 1 and 20. Methods 1 and 2 do not reach the required accuracy for any of the tested number of models, therefore, they will not be considered further. For methods 4, 5 and 6, 8 models are sufficient to fulfil the requirements at all tested points, and for Method 3, 20 models yield good accuracy for most of the points, but low MAC values for a few points. Adding more models does not improve the results in terms of the MAC and increases the calculation times, therefore the 20 model option is selected as the most efficient for Method 3.

Figure 2 shows the spread of the results when 20 models are used for Method 3, and 8 models are used for Methods 4, 5 and 6. It can be seen that Method 3 is still faster than methods 4 and 5, even if more models are included in the base. However, Method 6 presents solving times on the same range as Method 3, and a good accuracy for all points; therefore, this seems to be the most efficient option regarding both accuracy and speed.

It should be noticed that the results between the single-model and the multi-model reduction experiments show quite different tendencies for Method 6, which was not working well for single-model reduction, but is the most efficient method for multi-model reduction. This indicates that the method benefits from having a wider initial sample of vectors to create an orthogonal base.
3.3 Optimization of a plate

In this section, the thickness of the plate elements is optimized for two different objectives with the full and the selected MMR reduced models, and the results are compared.

3.3.1 Optimization of modal frequencies

As a starting point, a plate with a constant thickness of 0.1 mm is considered. The structure-acoustic interaction in this case is strong, which means that the modal frequencies for the uncoupled and the coupled system are quite different. The optimization problem consists in modifying the thickness of the plate so that the modal frequencies below 10 kHz are the same as for the uncoupled plate of 0.1 mm of thickness. In other words, the influence of the acoustic domain in terms of modal frequencies should be removed.

The objective function is defined as

$$g(\mathbf{t}) = \sum_{i=1}^{N} \left( \omega_{i}^{c}(\mathbf{t})^{2} - \omega_{i}^{u} \right)^{2}$$  \hspace{1cm} (35)

where $\mathbf{t}$ is the vector of thicknesses of the elements, $N$ is the number of modal frequencies below 10 kHz, $\omega_{i}^{c}$ is the $i$-th modal frequency of the coupled plate and $\omega_{i}^{u}$ is the $i$-th modal frequency of the uncoupled plate with 0.1 mm thickness. The gradient of the objective function is

$$\frac{dg(\mathbf{t})}{dt_{j}} = 2 \left( \omega_{i}^{c}(\mathbf{t})^{2} - \omega_{i}^{u} \right) \left[ \psi_{L}(\mathbf{t}) \right]^{T} \frac{\partial S(\mathbf{t})}{\partial t_{j}} [\psi_{R}(\mathbf{t})],$$  \hspace{1cm} (36)

where $t_{j}$ is the thickness of the $j$-th element, and

$$\frac{\partial |S(\mathbf{t})|}{\partial t_{j}} = \left( \frac{\partial |K(\mathbf{t})|}{\partial t_{j}} - \omega_{i}^{c}(\mathbf{t})^{2} \frac{\partial |M(\mathbf{t})|}{\partial t_{j}} \right).$$  \hspace{1cm} (37)

Therefore, it is required from the reduced model that the modal frequencies and the left and right modal vectors are accurately calculated.

The constrained optimization algorithm implemented in the Matlab Optimization Toolbox [16] function fmincon is used for the optimization (a detailed description of this method can be found in Ref. [17]). The analytical expression of the gradient is supplied, and the Hessian is approximated by a quasi-Newton method.

Each objective function evaluation takes on average 5.2 s when using the full system, 3.4 s for the system reduced with Method 3, 3.8 s with Method 4 and 3.5 s with Method 6. The resulting thickness designs when using the full system, and the reduction methods 3, 4 and 6 are shown in Figure 3. It can be seen that the
designs resemble much each other, and the mean errors between the results of the reduced methods and the full system are $1.76 \cdot 10^{-5}$ for Method 3, $1.27 \cdot 10^{-5}$ for Method 4 and $1.72 \cdot 10^{-5}$ for Method 6, which shows that the accuracy of the three methods is in the same range.

### 3.3.2 Optimization of compliance

In this case, the compliance of the system is to be maximized when the plate is excited by a unitary point force on its central point and the sum of all thicknesses is constrained below 0.05 mm. The objective function is

$$g(t) = \sum_{k=1}^{K} |u(k,t)^T f|$$

where $u$ is the vector of displacements and pressures, $f$ is the input force vector, $t$ is the vector of thicknesses, $k$ is the $k$-th frequency line and $K$ is the total number of frequencies included in the optimization, in this case, 100 frequency lines sampled logarithmically between 100 Hz and 10 kHz. The gradient can be calculated as

$$\frac{dg(t)}{dt_j} = \sum_{k=1}^{K} -\text{sgn}(u(k,t)^T f)u(k,t)^T \frac{\partial[S(k,t)]}{\partial t_j} u(k,t)$$

with

$$\frac{\partial[S(k,t)]}{\partial t_j} = \left( \frac{\partial[K(t)]}{\partial t_j} - \omega_k(t)^2 \frac{\partial[M(t)]}{\partial t_j} \right).$$

Therefore, in this case the accuracy of the displacement values is the main requirement from the reduced systems.

Each objective function evaluation takes on average 50 s when using the full system, 6.7 s for the system reduced with Method 3, 7.5 s with Method 4 and 7 s with Method 6. Figure 4 shows the designs obtained with the full and the reduced systems. The designs show larger differences than in the case of optimization of modal frequencies. This is due to the fact that the accuracy of the displacement vectors is lower than the accuracy of the modal frequencies, and even though the error is small, the deviations propagate through the optimization steps and result in significant differences in the result. However, there are many solutions to this optimization problem; therefore, the small deviations on the sensitivities lead to different results, which are not necessarily wrong. In this case, the four designs are quite different, but yield a similar final value of the compliance. If the objective function had a unique minimum, all methods would probably converge towards the same point eventually.

### 4. CONCLUSIONS

In this paper, several model reduction methods have been compared for a model of a plate coupled to an air column. For single-model reduction, using left and right modal vectors has been found to be the fastest approach. However, using only one kind of modal vectors and separating the acoustic and structural fields in the reduction base has shown a better accuracy. When combining both approaches, a poorer performance has been achieved. On the other hand, for multi-model reduction, the combined approach has performed more efficiently than the rest of the methods. All methods have been found to be accurate enough when optimizing the thickness of a plate to match certain modal frequencies. When optimizing the compliance, different designs have been reached by the different methods due to small deviations of the displacement vector leading to different paths in the optimization process. However, all designs are optimized for a reduced compliance; therefore, the results would be usable in any case. The multi-model reduction approach has been shown to be efficient for the presented cases, where the calculation time of the optimizations is dramatically reduced when using the reduced systems.
Figure 3: Thickness designs obtained with the full system and the three reduction methods for the optimization of modal frequencies.

Figure 4: Thickness designs obtained with the full system and the three reduction methods for the optimization of the compliance.
REFERENCES


