Detection of the dominant acoustic modes emitted by turbomachinery using compressed sensing

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ABSTRACT

Due to the aerodynamic interaction of turbomachinery components, as e.g. rotor and stator in axial turbomachinery or rotor and casing tongue in radial compressors, in a lot of cases only a small sub-set of acoustic modes is excited in the connected flow ducts. The knowledge of the dominant duct modes provides insight into the generating mechanisms and allows the calculation of the sound power emitted by the turbomachine. In this paper a method is presented, which allows the detection of the dominant modes with reduced efforts compared to standard methods. Typically, for the determination of the azimuthal mode orders Discrete Fourier Transformation or Least Squares Fit are used. For these methods, the amount of microphones needed for an exact reconstruction is based on the Nyquist criterion. The introduced method, Compressed Sensing, has the ability to reconstruct the dominant modes for underdetermined systems on the basis of non-equidistant microphone arrays. The reconstructed solutions of synthetic mode fields via compressed sensing are compared with Discrete Fourier Transformation and Least Squares Fit. Also, in practical applications microphone defects occur and is a circumstance that is accounted for in the presented study concluding the advantages and disadvantages of all three methods presented.

Keywords: Compressed Sensing, Dominant acoustic modes  I-INCE Classification of Subjects Number(s): 74.9

1. INTRODUCTION

At subsonic flow conditions axial and radial turbomachinery predominantly excite tonal components at the harmonics of the blade passing frequency (BPF). The mode content of the tonal components depends on the particular source mechanisms. In axial turbomachinery the aerodynamic interaction of rotor and stator typically causes a few dominant azimuthal mode orders (1), (2), which was first formulated by Tyler and Sofrin (3). Compared to this the situation in radial turbomachines is manifold. Radial compressors equipped with vaned diffusors radiate dominant modes into the intake pipe that comply with the rule of Tyler and Sofrin, as was found e.g. by Raitor (4). In a design with a vaneless diffuser, however, the generated mode spectrum looks different, since the dominant noise source results from the interaction of the rotor wakes with the tongue of the volute casing. It is the objective of a current project to determine the sound power that is emitted by a radial compressor into the outlet pipe. Since the measurement effort should be kept small, i.e. the number of microphone positions should be much lower than the total number of modes propagating in the flow duct, the challenge consists in a good estimation of the contribution of the dominant modes.

It has been shown by Huang (5) and Behn et al. (6) that in the case of under-sampling the application of Compressed Sensing for a decomposition of the ducted sound field into the azimuthal mode constituents has an advantage compared to Discrete Fourier Transformation (DFT) and Least Squares Fit (LSF). Compressed sensing is typically applied to underdetermined linear systems with the prior knowledge, that the true solution is sparse. Hayashi et al. (7) give a good overview of existing algorithms. The OMP algorithm given by Tropp et al. (8) picks the dominant components in a sparse system successively and is efficient, fast and easy to implement. It further implies the deconvolution of the measured pressure signal regarding the dominant mode contributions, which is used by Behn et al. (6) in an enhanced OMP version for the estimation of the remaining (non-dominant) mode components. Behn et al. (6) further introduced different approaches for the design of optimal sensor arrays.

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For the current study a non-uniform azimuthal sensor array has been optimized using an approach of Behn et al. (6), which evaluates the mutual coherence of the sensing matrix. The quality of the array for azimuthal mode detection using Compressed Sensing is assessed by application to synthetic test data and the results are compared with the DFT respectively LSF based methods. Further comparisons are performed with a uniform array, which consists of equidistantly spaced sensors and usually is employed in test rigs, and another non-uniform array, for which the azimuthal sensor positions were are randomly chosen on basis of a stochastic uniform distribution. With regard to the practical application it is an important aspect, how the analysis results achieved with the different arrays and methods are impacted by microphone defects. Thus a systematic study is carried out, in which the defect microphone positions are randomly varied and in addition the number of microphone failures is successively increased.

2. METHODS TO DETERMINE AZIMUTHAL MODE SPECTRA

Considering time harmonic waves the pressure field in a ducted flow can be derived from the general solution in modal terms of the convective Helmholtz equation in cylindrical coordinates

\[ p(x, r, \Theta) = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \left( A_{mn} e^{ik_{mn}r} + A_{\bar{m}n} e^{ik_{\bar{m}n}r} \right) f_{mn}(r) e^{im\Theta} . \]  

(1)

This equation holds the up- and downstream axial wavenumbers and complex amplitudes of the radial modes with azimuthal mode order \( m \) denoted by \( k_{mn}, k_{\bar{m}n}, A_{mn} \) and \( A_{\bar{m}n} \) respectively. The radial mode shapes are denoted by \( f_{mn}(r) \) and can be derived numerically or analytically on basis of the duct geometry, frequency and the specific inhomogeneous or homogeneous flow respectively. This also applies for axial wavenumbers. In the detailed analysis of turbomachinery noise, often the first analysis step consists in an azimuthal mode decomposition of the sound field in the flow duct (1). Denoting the value of the sum over all radial mode orders \( n \) for fixed azimuthal mode order \( m \) as the complex azimuthal mode amplitude \( A_{m} \), the decomposition of the azimuthal sound pressure distribution in azimuthal modes at a fixed axial and radial position yields

\[ p(\Theta) = \sum_{m=-\infty}^{\infty} A_{m} e^{im\Theta} . \]  

(2)

Due to the decomposition being based on measurements with wall flush-mounted sensors, the radial dependency is neglected in the above formulation. Based on the Fourier transform at a fixed axial position \( x \) the azimuthal mode spectrum of the sound pressure with respect to the azimuthal coordinates is determined by

\[ A_{m} = \frac{1}{2\pi} \int_{0}^{2\pi} p(\Theta_{k}) e^{-im\Theta_{k}} d\Theta_{k} . \]  

(3)

In an experimental context sensors at discrete positions are applied, so that the integral in Eq. (3) is replaced by a sum over the sensors at discrete positions \( \Theta_{k} \) yielding the expression for the Discrete Fourier Transform (DFT) described by

\[ A_{m} = \frac{1}{K} \sum_{k=1}^{K} p(\Theta_{k}) e^{-im\Theta_{k}} . \]  

(4)

The relation between the complex sound pressure values \( p \) measured at \( K \) sensor positions and the underlying azimuthal mode spectrum \( a \), with \( M \) propagating modes being considered, can be formulated as a system of linear equations as

\[ p = Wa . \]  

(5)

Here \( W \) denotes the sensing matrix of size \( KxM \) with the mode functions evaluated for the \( m \)th mode order at the \( k \)th sensor position as entries. In this study the modal range is always symmetrical w.r.t. \( m = 0 \), i.e. \( m \in \{-m_{\text{max}}, \ldots, 0, \ldots, m_{\text{max}}\} \) with a total number of modes \( M = 2m_{\text{max}} + 1 \). The azimuthal mode analysis is performed by solving the system of linear equations in Eq. (5).

2.1 Conventional Methods

The most commonly applied methods to determine the azimuthal mode spectrum are a least-squares fit (LSF) and the Discrete Fourier Transform (DFT). Applying the LSF to Eq. (5) yields in matrix-vector notation:

\[ a = [W^{H}W]^{-1}W^{H}p, \]  

(6)

where the superscript \( H \) denotes the adjoint matrix. The term \( [W^{H}W]^{-1}W^{H} \) is called the pseudo-inverse of matrix \( W \). Applying the pseudo-inverse to the sound vector \( p \) results in a mode amplitude vector, which minimises
the l2-norm of the residuum according to the cost function \( J = \| e \|_2 = \| W a - p \|_2 \). In matrix-vector notation the DFT takes the form
\[
a = \frac{1}{K} W^H p.
\] (7)

In the standard application with sensors being equally spaced along the circumference, following the Nyquist-Shannon theorem the number of sensors has to be larger than the number of propagating modes \( M \). In this case, the system of linear equations in Eq. (5) is overdetermined and DFT respectively LSF perform well. However, if the sound field is subsampled, then typically aliasing effects occur, which can hamper the interpretation of the mode spectrum. The aliasing effects can be avoided by arranging the sensors with non-uniform spacing, but with drawback of the occurrence of side lobes even for oversampled sound fields. Rademaker et al. (9) proposed a cost function that can be used in an optimisation of non-uniform arrays. It minimises the mean square of the side lobe amplitudes for the analysis of a number of modes \( M \) larger than the number of sensors \( K \).

2.2 Compressed Sensing

Compressed Sensing is an approach for solving underdetermined systems of linear equations under the assumption that the solution vector is sparse, i.e. in this case the azimuthal mode spectrum consists only of a few dominant modes. In such cases, it is assumed that the mode spectrum is compressible, i.e. the sound pressure pattern at the microphones is well-approximated by the contribution of the dominant modes.

Given the sparsity of the mode amplitude vector, the correct solution can be found by minimising the l1-norm of the solution vector (10), (11). In terms of azimuthal mode analysis this can be formulated mathematically as
\[
a = \arg\min_{a \in \mathbb{C}^M} \| a \|_1 \text{ subject to } \| p - W a \|_2 \leq \varepsilon.
\] (8)

The quantity \( \varepsilon \) represents the noise that is inherent in the system. The minimisation problem can be solved in various ways. The algorithm used in this paper is an extension of the OMP algorithm that was firstly applied in the context of Compressed Sensing by Tropp and Gilbert (8). The OMP algorithm searches for the dominant modes in an iterative process until a stop criterion is reached. This stop criterion is based on a minimum residuum or a pre-defined amount of iteration steps. In each of these iteration steps the highest dominant mode is determined. Subsequently this mode is deconvolved from the sound pressure vector and the iteration process continues. Therefore the amount of the reconstructed modes is equal to the number of iterations.

Eldar and Kutyniok (12) mention that sparse recovery by minimisation of the l1-norm is most likely successful, if the column vectors of the system matrix \( W \) are incoherent. The mutual coherence \( \mu \) of the matrix \( W \) is defined as:
\[
\mu(W) = \max_{1 \leq i < j \leq M} \frac{\| \langle \omega_i, \omega_j \rangle \|_2}{\| \omega_i \|_2 \| \omega_j \|_2}
\] (9)

where \( \omega_i \) denotes the \( i \)-th column vector of \( W \). Since the system matrix \( W \) only depends on the modal range and the positions of the sensors in the microphone array, the mutual coherence can be regarded as a property of the sensor array. For a configuration with \( K \) microphones and \( M \) modes, the mutual coherence lies within the range
\[
\mu(W) \in \left[ \sqrt{\frac{M - K}{K(M - 1)}}, 1 \right].
\] (10)

The lower bound is called Welch bound and depends only on the size of matrix \( W \). A small value of the mutual coherence indicates that any pair of column vectors are close to orthogonal. The error between the exact solution and the calculated mode spectrum is smaller the lower the mutual coherence of an array is. This is true, if the number of determined mode amplitudes \( s \) fulfills the inequality
\[
s < \frac{1}{\mu(W)}.
\] (11)

Furthermore, the mutual coherence is linked to the side lobe level (SLL) and can be determined via (6)
\[
SLL = 20 \log_{10} \left( \frac{1}{\mu(W)} \right).
\] (12)

Due to the dominant contribution being deconvolved when using the OMP algorithm, Behn et al. (6) exploited this feature to estimate the remaining mode spectrum by applying an enhanced version of the OMP algorithm. This algorithm is used in the course of this study when referred to Compressed Sensing. Knowing the mutual coherence of a sensing matrix gives the maximum amount of sparse signals that can be accounted for in a signal reconstruction using the OMP algorithm. In the iterative procedure of the OMP-algorithm the sound pressure
vector is successively deconvolved with the determined mode amplitudes. After the OMP algorithm has terminated
the remaining mode spectrum is estimated by applying the DFT or LSF on the residual sound pressure vector. The
choice of the method to use in this step is determined by the still remaining system of linear equations on basis of
the condition number of the system matrix. A comparison of OMP to the enhanced OMP is given in Behn et al.
(6).

The mutual coherence can be used as a criterion for the optimisation of a sensor array. Behn et al. (6) mention
two methods. The method applied here was firstly described by Xia et al. (13). It is based on the postulation that
the so-called Welch-Bound array consisting of K sensors is a subset of a uniform array consisting of M sensors.
From this initial uniform array random subsets are created and the array with the best coherence is determined in
an iterative process. This method has been used in this study to create an optimised non-uniform sensing array.

3. IMPACT OF MICROPHONE DEFECTS ON THE MODE RECONSTRUCTION

Reconstruction of the azimuthal mode structure on basis of an underdetermined system leads to errors when
applying the standard methods DFT and LSF. These methods suffer from the occurrence of sidelobes, which in
return create artificial contributions into the mode spectrum. Hence, determining the dominant modes of a signal
can become impossible. Compressed Sensing on the other hand shows promising results for the reconstruction
of dominant modes when applied to underdetermined systems (6), (5). The effects that a successive reduction of
microphones from an initial setting has on the reconstruction quality are studied using DFT, LS and Compressed
Sensing.

In the current study no more than \( K = 16 \) microphones shall be used. Three different sensor array designs
are considered: (i) uniform, (ii) randomised non-uniform, (iii) optimised non-uniform. The randomised non-
uniform sensing positions are determined via an uniformly random distribution whereas the optimised non-uniform
sensing positions are determined on basis of the aforementioned optimisation algorithm. The optimisation has been
conducted for a system with \( K = 16 \) microphones and a maximum mutual occurrence of azimuthal modes of \( M = 31 \) and
the corresponding Welch-Bound array has been determined. To assess the effects that the reconstruction
methods have in over- and under-determined systems, a test case comprising in a total of \( M = 15 \) azimuthal modes
has been chosen. Assuming microphone defects in the range of \( N_D \in [0 : 6] \) the system undergoes a transition from
an over- into underdetermined system at \( N_D = 2 \). Depending on the positions and amount of defects, different
sidelobe patterns can occur in the reconstruction process (14). In order to get a representative overview the defects
were varied using a Monte Carlo algorithm.

3.1 Coherence of Sensing Matrices

The coherences of all three sensing matrices resulting for the uniform, randomised non-uniform and optimised
non-uniform microphone array are determined via Eq. (9). The effects of microphone defects at random positions
are studied in a Monte Carlo study with 500 iterations. In Figure 1a the mutual coherence is plotted for each of the
three arrays with the amount of microphone defects \( N_D \in [0 : 6] \) varying along the x axis.

![Figure 1: Mutual coherence of the sensing matrices varying with microphone defects in an uniform, randomised
non-uniform resp. optimised non-uniform microphone array consisting of K=16 microphones applied to an
azimuthal mode range of \( m \in [-7 : 7] \) and \( m \in [-16 : 16] \) resulting in \( M = 15 \) and \( M = 33 \) azimuthal modes,
respectively. 500 different sets of microphone defects were randomly chosen in a Monte Carlo study.](image)
Table 1: Maximum sparsity value $s_{ND}$ based on the averaged coherence of each sensing matrix resulting from the Monte Carlo study

<table>
<thead>
<tr>
<th></th>
<th>$s_0$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
<th>$s_5$</th>
<th>$s_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>uniform</td>
<td>15</td>
<td>8</td>
<td>5</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>opt. non-uniform</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>rdm. non-uniform</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

Due to the coherence being studied for microphone defects at uniformly random chosen positions in each iteration step, a variety of solutions are derived. The statistical population of the resulting samples are plotted in boxplots. Boxplots are divided into quartiles. The first quartile, being the lowest margin of the box, splits off the lower 25% of data, the second cuts the data set in half around the median and the third quartile, being the highest margin of the box, splits off the upper 25% of the data. Outliers can occur outside a upper and lower limit that denote extremities, but are not present in this specific data. The coherence of the initial sensing matrix of the uniform sensing matrix is zero, followed by the optimised non-uniform at $\mu \approx 0.18$ and the randomised non-uniform array with a coherence of $\mu \approx 0.43$. Applying the test case for the microphone defaults at randomly chosen positions shows, that in all three cases the coherence increases. Furthermore the Monte Carlo study shows fluctuations of the mutual coherence for all three arrays, concluding that the positions of the microphone defects have an impact on the mutual coherence and therefore on the reconstruction precision. The transition from an overdetermined system at $N_D = 2$ has no drastic effect on the sensing matrix. Hence, the test case shows that an uniform array is the most incoherent followed by the optimised and subsequently the randomised array.

The maximum sparsity of a signal that can be reconstructed via Compressed Sensing can be determined on basis of Eq. (11). The amount of sparse signals was calculated from the averaged coherence resulting in the Monte Carlo study for the different arrays and variable number of sensor defects, see Table 1. The amount of sparse signals to each microphone defect is denoted by $s_{ND}$.

Note, that if the applied modal range exceeds the modal range used to construct the uniform array, aliasing occurs and the mutual coherence results in a value of $\mu = 1$. As can be seen in Figure 1b, this also applies for the optimised non-uniform array, due to the optimisation procedure determining the array from an initial uniform array. Obviously the amount of azimuthal modes has a small effect on the mutual coherence of the randomised array.

Due to the sensing matrix with the uniform and optimised non-uniform array having the best coherences and the azimuthal range chosen in the following test case not exceeding the range of $M = 32$ (no aliasing effects can occur), the reconstruction analysis of a synthetic signal using these two arrays is presented here.

3.2 Reconstruction of Azimuthal Mode Spectra

A full reconstruction of a pressure field is performed here with DFT, LSF and Compressed Sensing. The measurements concluding the pressure vector are simulated by using the uniform and optimised non-uniform sensing array. This vector field is created by assigning an azimuthal mode spectrum with pre-defined amplitudes for three dominant and a superposed uniform random amplitude spectrum for the remaining modes. These random amplitudes are distributed around a mean value of $\tilde{A}_{\text{rand}} = 1 \cdot 10^{-2}[\text{Pa}]$ (with a deviation of the same value as $\tilde{A}_{\text{dev}} = 1 \cdot 10^{-2}[\text{Pa}]$) in a range of $m_{\text{max}} = 7$ concluding a total of $M = 15$ azimuthal modes. The pre-defined dominant modes $A_m$ at specific azimuthal positions $m$ are presented in Table 2.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$-5$</th>
<th>$-3$</th>
<th>$1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_m[\text{Pa}]$</td>
<td>$2 \cdot e^{i\pi/5}$</td>
<td>$0.2 \cdot e^{2i\pi/5}$</td>
<td>$0.8 \cdot e^{i\pi/3}$</td>
</tr>
</tbody>
</table>

Using this modal spectrum and the aforementioned microphone positions a pressure field vector is created by applying Eq. (5). To ensure the resulting pressure field vector is an exact solution of the predefined spectrum, an overdetermined sensing matrix $W$ is used by applying $K = 16$ microphones to the uniform and optimised non-uniform sensing positions. To determine the quality of the reconstruction methods this pressure field vector is subsequently reconstructed using DFT, LSF and Compressed Sensing whilst accounting for a range of microphone defects at varying positions.

In Figure 2 the original spectrum and the reconstruction with DFT, LSF and Compressed Sensing (CS) with five microphone defects applied to the initial uniform sensing matrix is plotted as an exemplary solution. This amount of microphone defects leads to subsampling of the modal field, through which the Nyquist criteria does not hold. Therefore the reconstruction via DFT and LSF shows artifacts that are superposed onto the synthesised original signal. This phenomenon is illustrated in the plots underneath the original spectrum of Figure 2 for DFT.
and LSF respectively. In all plots the averaged random amplitudes of the original spectrum is denoted by a solid black line and is computed by

$$SPL_{A\,rand} = 10\log_{10} \left( \frac{1}{M_{\text{rand}}} \sum_{m=m_{\text{max}}}^{m_{\text{min}}} 10^{SPL_{rand}/10} \right).$$

Here $SPL_{A\,rand}$ denotes the averaged sound pressure level of all azimuthal modes without the dominant modes. Therefore the summation is conducted over all modes subtracting the dominant ones. In this case the denominator
then results to $M_{\text{rand}} = M - 3$ and $\text{SPL}_{\text{rand}}$ denotes all azimuthal sound pressure levels without the three dominant ones. The solution of the averaged amplitudes after the reconstruction and again not considering the dominant modes is shown by the red dashed line and is also derived from Eq. (13). Hence, one must interpret the averaged non-dominant mode spectrum in two ways. Firstly the original non-dominant mode spectrum is a representation of actual physical noise that occur in ducted turbomachine acoustics and are a result of uncorrelated sound sources, e.g. turbulences. Secondly the reconstructed non-dominant mode spectrum is a result of spurious occurrences and is a mathematical problem. The averaged random amplitudes after reconstruction with DFT and LSF is at a level of about 85 dB and LSF reaches 81 dB, whereas the initial original spectrum holds a value of 58 dB. This concludes that the signal-to-noise ratio between the dominant mode and the mean of the azimuthal modes without the dominant ones as derived by Eq. (13) becomes much smaller in the reconstruction with DFT and LSF. Here the terminology signal-to-noise ratio is used as a reference to the highest dominant mode to the averaged amplitudes without the pre-defined dominant modes. Due to spurious artifacts, the second and third dominant mode fall in the SPL range of these artifacts and therefore cannot be determined any more. Finally the reconstruction with Compressed Sensing shows an exact reconstruction of the dominant modes without creating a highly overestimated noise spectrum. The mean of the estimated non-dominant azimuthal modes when reconstructed with the enhanced OMP algorithm is an exact match to the original one, although single non-dominant modes differentiate from the original ones.

An exemplary solution resulting from the Monte Carlo study with five microphone defects and for the reconstruction on basis of the optimised non-uniform sensing matrix is given in Figure 3. Here the same three dominant modes with a different randomised spectrum for the non-dominant modes to the before conducted case is applied defining the original spectrum. All aforementioned findings apply for this case with only a few differences. One difference to the aforementioned is, that the reconstructed mean of the non-dominant modes using the enhanced OMP algorithm overestimates the original by a few dB. Also, in contrary to the DFT solution the third dominant azimuthal mode at mode order $m = -3$ seems to be reconstructed accurately when applying LSF in this specific case. Note that in both cases, the result shows that even though the condition of the sparsity level being infringed on basis of the coherence of the sensing matrix, the reconstruction of the third dominant mode was conducted successfully. The effect that the reconstruction via DFT, LSF has on the dominant modes and the signal-to-noise ratio are compared with the applied aforementioned features and are presented in the following sections.

3.2.1 Reconstruction Error on Dominant Modes

The primary goal of this section is to determine the bandwidth of the reconstruction error through the variation of amount and position of sensors. The SPL values of the reconstructed dominant modes are compared with the original ones in a Monte Carlo study with 500 iterations for the initial uniform and optimised non-uniform sensing positions. The reconstruction error of each dominant mode is determined individually by

$$e = \frac{(|A_e| - |A_r|)}{|A_e|} \cdot 100\% \quad (14)$$

with $A_e$ and $A_r$ denoting the exact and reconstructed complex amplitudes of each dominant mode respectively. The data gathered for the first three dominant modes for both sensing arrays are plotted in Figures 4, 5 and 6. Note, that in the case that an underdetermined system is used the reconstruction via DFT and LSF cannot reconstruct all dominant modes due to the rise of the overall sound pressure level of the remaining modes. This has been shown in the section before. Nonetheless, in this artificial test case the dominant modes are known from the start and therefore the comparison of these can be made using Eq. (14).

In Figure 4 the reconstruction error of the strongest dominant mode for both arrays, each method and the amount of microphone defects is depicted. The reconstruction error produced when applying the methods to the uniform sensing case is presented in Subfigure 4a. The initial setting without a microphone defect is exact due to the difference of the reconstructed and original dominant mode being at a value of zero. The reconstruction using DFT produces a error when one microphone defect occurs due to the array then becoming non-uniform. By a further increase of microphone defects, the mean error stays constant around zero but the statistical population of the data increases. Single outliers present are illustrated as single points outside the whiskers of the boxplots and represent exceptions that occur with a relative frequency. These findings indicate, that the position of the microphone defects have an influence on the reconstruction quality. This applies for all further cases. The reconstruction via LSF shows an increase of the reconstruction error after the transition point from over- to under-determined is reached at $N_D = 2$. Similar effects apply for the reconstruction methods used for the optimised non-uniform arrangement of the sensing positions illustrated in Subfigure 4b. The difference of this solution is that the DFT reconstruction has an initial error without any microphone defects. This is expected due to the sidelobes that occur when DFT is applied to non-uniform arrays even when overdetermined systems are used. Here the same findings apply as for the uniform array, due to the mean of this error staying constant and the range of the error increasing with an increase of microphone defects. This deviation is a result of interference of the azimuthal mode at position $m = -5$ and the...
sidelobes of the remaining modes, particularly with the other dominant modes Huang et al. (5). The reconstruction of the dominant mode using the Compressed Sensing algorithm in both cases is exact with only a small deviation occurring. In all solutions of the reconstructions the varying statistical population denoted by the boxplots are a result of the microphone defaults occurring at different azimuthal positions. These are chosen at random positions in each iteration step of the Monte Carlo study.

A comparison of the reconstructed second strongest with the original complex amplitude at azimuthal mode order \( m = 1 \) is plotted in Figure 5. The overall error range for both sensing cases presented in Subfigures 5a and 5b clearly increases when applying DFT or LSF compared to the reconstruction solutions of the first dominant mode. Compressed Sensing again gives an exact reconstruction with a minimum error occurring due to the increase of defects at certain microphone positions. The reconstruction via LSF results in a constant increase of error due to an increase of microphone defects. The mean error is equal to the error produced during the reconstruction of the strongest dominant mode i.e. at \( N_D = 6 \). Hence, the resulting mean error due to LSF reconstruction of the dominant modes seems to be constant. The difference here is that the positions of the microphone defects have a larger impact on the overall range of the reconstruction error. This can be seen in the outliers that reach an maximum of around \( e \approx -100\% \) at \( N_D = 6 \). The same applies for DFT with the mean again being constant around a value of zero but an increase of the error range. The mean error created by the reconstruction via DFT on the non-uniform array is constant at a value around the initial error resulting from the non-uniformity at \( N_D = 0 \) and therefore is larger than the DFT reconstruction conducted on the uniform grid. In contrary to the initial uniform sensing array, the deviation in this case is lower. The reconstruction using LSF and Compressed Sensing on the non-uniform sensing positions has no significant different effect on the reconstruction quality in comparison with
the initial uniform sensing positions.

![Diagram showing error for different settings]

(a) Initial uniform sensing positions
(b) Optimised non-uniform sensing positions

Figure 6: Error of the third strongest dominant azimuthal mode at azimuthal mode order \( m = -3 \).

The same comparison is made for the third strongest dominant mode at azimuthal mode order \( m = -3 \) and is plotted in Figure 6. Again the overall range of the error in both cases clearly increases for all three methods applied. Compressed Sensing gives the best result having a minimal error with a small increase due to an increase of microphone defects. In contrary to the before findings, the mean error of the reconstruction via LSF is lower than DFT for both cases as shown in Subfigures 6a with 6b. Furthermore the LSF reconstruction error differentiates slightly when applied to the different initial sensing positions and amount of microphone defects. Using different initial sensing arrays has only a small influence on the reconstruction error using DFT for the lowest dominant mode, due to the mean error produced by DFT being similar for both cases. A higher difference lies at \( N_D = 0 \) due to DFT being applied on a non-uniform array presented in Subfigure 6b.

### 3.2.2 Signal-to-Noise Ratio

Here the signal-to-noise ratio is referred to as the deviation of the dominant mode to the mean azimuthal remaining spectrum that does not account for the three dominant modes. The effect of microphone defects on the signal-to-noise ratio of the highest dominant mode to the mean reconstructed non-dominant complex amplitude spectrum is studied in the following and is determined by

\[
SNR = \frac{SPL_{A_{\text{dev}}}}{SPL_{A_{\text{rand}}}}.
\]

The index \( A_{\text{dev}} \) denotes the sound pressure level of the highest dominant mode and the averaged sound pressure level of the remaining modes is computed via Eq. (13) and is denoted by \( A_{\text{rand}} \).

Figure 7 illustrates the signal-to-noise ratio resulting from applying the reconstruction methods at hand to the measurement vector when applied to the initial uniform and optimised non-uniform array. Multiple microphone defects at varying positions are considered in the evaluation. Subfigure 7a and 7b denote the solutions for the uniform and the optimised non-uniform array respectively. Note, the range of the statistical population that occur are attributed to the non-dominant azimuthal modes being created new in each iteration step of the Monte Carlo study in an uniformly random distribution. Hence, an initial deviation resulting from \( \Lambda_{\text{dev}} = 1 \cdot 10^{-2}[Pa] \) is present. Considering no microphone defects the uniform positioning determines the signal exactly, due to the deviation simply being a result of the aforementioned implementation. When observing the solution produced by the reconstruction using DFT an exact reconstruction is made when applied to the initial overdetermined uniform setting. As soon as a microphone defect occurs artifacts are created that superpose on the actual signal. Hence, the signal-to-noise ratio of the dominant mode to the noise spectrum decreases due to an increase of the overall remaining mean azimuthal complex amplitudes. LSF on the other hand can reconstruct the signal in an exact manner independent of the array used as long as an overdetermined system is given. At the transition from an over- to underdetermined measurement system, the same artifacts as for the DFT and therefore a reconstruction creating spurious effects that are superposed onto the original data occur. Here also the signal-to-noise ratio decreases. Throughout all cases, Compressed Sensing using the enhanced OMP algorithm keeps the signal-to-noise ratio constant, concluding its effectiveness in signal reconstruction for underdetermined non-uniform systems.
4. COMPRESSED SENSING WITH REDUCED SENSOR COUNT

Compressed Sensing shows its effectiveness when applied to azimuthal mode reconstruction with a lowering number of microphones. To determine the optimal amount of microphones for the aforementioned test case, three optimised arrays are designed and applied here. The test case comprises the same dominant modes as before but with a different spectrum for the non-dominant modes. The mutual coherence and the sparsity of each sensing matrix is a result of the optimisation algorithm for \( K = 8 \), \( K = 6 \) and \( K = 4 \) microphones and an azimuthal range of \( m \in [-7 : 7] \). The positions of the microphones are illustrated in Figure 8. The purple, green, blue and gray points at the azimuthal positions denote the non-uniform arrays with \( K = 4 \), \( K = 6 \), \( K = 8 \) and the uniform array with \( K = 16 \) respectively. Furthermore the pressure field resulting from the pre-defined azimuthal spectrum is depicted in this Figure. The mutual coherence that are close to the Welch-Bound array and the amount of sparse signals that can be reconstructed as a result of the aforementioned optimisation algorithm are given in Table 3.

Figure 8 illustrates the exact signal and the reconstruction using the enhanced OMP algorithm for all three optimised arrays. The signal using \( K = 8 \) microphones gives an exact reconstruction of the dominant modes, with the non-dominant modes being estimated with clear deviation towards the original signal. Using Compressed Sensing with \( K = 6 \) microphones at optimised sensing positions results in an increasing error that is superposed...
onto the dominant modes. This is a result of an increasing mutual coherence and therefore the reconstruction quality being adversely affected due to the occurrence of higher level sidelobes, i.e. Eq. (12). This also applies for the case using $K = 4$ microphones. In this case a further increase of the sound pressure level of the reconstructed non-dominant modes can be observed due to the increasing sidelobe level. Hence, the lowest amount of microphones to reconstruct a signal is limited. In this specific case, $K = 8$ microphones should be used for an exact reconstruction of the dominant modes.

Table 3: Coherence and maximum sparsity of three optimised sensing matrices for $K = 8$, $K = 6$ and $K = 4$ microphones.

<table>
<thead>
<tr>
<th>$K$</th>
<th>$\mu$</th>
<th>$s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>0.25</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>0.4</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>0.5</td>
<td>2</td>
</tr>
</tbody>
</table>

Figure 9: Azimuthal mode spectrum of original and reconstructed spectrum using the enhanced OMP algorithm with varying amount of microphones.

5. CONCLUSION AND OUTLOOK

The performance of azimuthal mode detection using uniform respectively non-uniform sensor arrays in combination with DFT, LSF and Compressed Sensing algorithms was examined by application to synthetic test data. The study showed that in over-determined systems, i.e. the number of sensor positions exceeds the number of azimuthal mode orders, DFT and LSF give an exact reconstruction of the sound field. If DFT is applied to a non-uniform microphone arrangement, side lobes may impact the quality of reconstruction by masking mode amplitudes of second-order dominance. LSF on the other hand can give an accurate reconstruction when applied to non-uniform sensing arrays. However, in an underdetermined system the LSF reconstruction produces artifacts, which lower the dynamic range of the analysis in a similar way as in the DFT.

In contrast to this, by application of the Compressed Sensing algorithm the dominant azimuthal constituents of the mode spectrum can be accurately reconstructed in principle independent of the array type (uniform or non-uniform) and for both over-determined and under-determined systems. A pre-requisite is that the mode spectrum is sparse. Nonetheless also the level of the weak modes can be well estimated with the enhanced version of the compressed sensing OMP algorithm. The sparsity level of the reconstruction, i.e. the number of dominant modes that can be reconstructed, is depending sensitively on the total number and the positions of the sensors in the array. Microphone defects show a much smaller impact compared to the outcome of the DFT and LSF algorithms.

Regarding the estimation of the sound power emitted by a radial compressor into the outlet pipe, the limits of the enhanced OMP algorithm and the performance of low count sensor arrays will be further investigated and
the analysis of mode spectra that exhibit no dominant components are planned. Beyond this an extension of the method to the determination of the radial mode content will be considered and the quality of the Compressed Sensing based mode decomposition will be assessed in specific experimental tests.

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REFERENCES