On the cost reduction of the fast BEM Hierarchical Matrix approach for partly symmetric surfaces

Boris Dilba[1]; Sören Keuchel[2]; Olgierd Zaleski[3]; Otto von Estorff[4]

[1] Novicos GmbH, Hamburg, Germany

[2] Novicos GmbH, Hamburg, Germany

[3] Novicos GmbH, Hamburg, Germany

[4] Institute of Modelling and Computation, Hamburg University of Technology, Germany

ABSTRACT

Sound radiation of vibrating structures is of interest in many engineering disciplines. When considering structures under free field conditions the sound radiation can advantageously be calculated by means of the Boundary Element Method (BEM). Compared to the Finite Element Method (FEM) the standard BEM has the drawback of fully populated system matrices. Fast Boundary Element Methods like the Fast Multipole Method or the Hierarchical Matrix approach effectively reduce this drawback.

For the H-Matrix approach rotational symmetric or symmetric structure surfaces can be exploited to further reduce the costs for assembling and storing the approximated system matrix due to its Toeplitz structure.

In this paper, it will be outlined how this reduction can also be applied to overall non symmetric surfaces but primarily composed of partly symmetric parts. In order to efficiently solve the linear system of equations while maintaining the advantage of reduced storing costs the usually applied hierarchical LU-decomposition for preconditioning the iterative solver is not recommendable. This is due to the requirement of the total approximated system matrix. Hence an alternative efficient preconditioner will be used for the exterior problem in this paper. The efficiency of the reduced matrix representation will be discussed on numerical examples.

Keywords: H-Matrix, symmetric, OSRC

1. INTRODUCTION

In the frequency domain the wave propagation is described by the Helmholtz equation

$$\Delta p(\boldsymbol{x}) + k^2 p(\boldsymbol{x}) = 0 \quad \boldsymbol{x} \in \Omega \quad k = {}^{\omega}/_{c}. \tag{1}$$

The wave number is given by $k = \omega/c$ with the speed of sound c and the angular frequency $\omega = 2\pi f$. Applying integration by parts at the weak formulation of (1) together with the property of the fundamental solution:

$$G(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{y}|} e^{-ik|\boldsymbol{x}-\boldsymbol{y}|},$$
$$\Delta G(\boldsymbol{x}, \boldsymbol{y}) + k^2 G(\boldsymbol{x}, \boldsymbol{y}) = -\delta(\boldsymbol{x}, \boldsymbol{y}) \tag{2}$$

leads to the Boundary Integral Equation (BIE):

[1] dilba@novicos.de
[2] keuchel@novicos.de
[3] zaleski@novicos.de
[4] estorff@tuhh.de

$$c(\boldsymbol{y})p(\boldsymbol{y}) + \int_{\partial\Omega} \frac{\partial G(\boldsymbol{x},\boldsymbol{y})}{\partial n(\boldsymbol{x})} p(\boldsymbol{x})\partial\Omega(\boldsymbol{x}) = i\rho\omega \int_{\partial\Omega} G(\boldsymbol{x},\boldsymbol{y}) v_n(\boldsymbol{x})\partial\Omega(\boldsymbol{x}), \tag{3}$$

where ρ denotes the density of the fluid, p the acoustic pressure and v_n the velocity in normal direction $\vec{n}$. According to the procedure of the collocation method equation (3) results in the linear system of equations for the discretised Helmholtz equation, cp. [1] and [2],

$$\left(\frac{1}{2}I + K\right)\boldsymbol{p} = i\rho\omega V \boldsymbol{v_n}. \tag{4}$$

For the conventional BEM the matrices K and V are dense and thus have quadratic memory requirements. Due to the fully populated system matrices solving eq. (4) has a complexity of $O(mn^2)$ to $O(n^3)$ with $m < n$. As a result the problem size of the conventional BEM is limited by memory requirements and solution time.

2. Hierarchical Matrix compression

The application of fast compression techniques push the computational limits of the BEM such that larger problems can be solved. One of the most popular methods is the Fast Multipole Method (FMM), which generates an approximation of the matrix-vector product with quasi linear complexity $O(n\,log(n))$. This can be achieved by approximating the kernel function by an analytic approach. Due to the fact that only the matrix-vector product is approximated by the FMM iterative solvers e.g. GMRES can be applied to solve eq. (4).

Another approach to reduce memory requirements and solution time is the concept of hierarchical matrices (H-Matrix), as described in [3] and [4]. The idea of the H-matrix approach is based on the compression of the system matrix. Due to the properties of the fundamental solution (2) and a suitable partitioning of the system matrix, submatrices exist with exponentially decreasing singular values [3], which can efficiently be approximated by a low-rank representation in outer-product form

$$A = \sum_{i=1}^{k} u_i v_i^H. \tag{5}$$

In order to generate a suitable matrix partition the set of degrees of freedom (dofs) $I = \{1, \dots, n\}$ will geometrically be subdivided into a cluster tree $T(I)$. Starting to subdivide the root $\tau = I$ of the cluster tree $T(I)$ into 2 geometrical separated son clusters $S(\tau) = \{\tau_1, \tau_2\}$ will generate the next level of the cluster tree $T(I)$. The cluster tree $T(I)$ will be generated recursively by applying $S(\cdot)$ to all clusters until a specified minimal cluster size is reached. Every level of $T(I)$ corresponds to a subdivision of I where the lowest level represents the leaves $\mathcal{L}(T_I)$ of the cluster tree. For the subdivision of the clusters the corresponding indices will be sorted according to the principal component analysis in order to establish a balanced cluster tree. Based on the cluster trees for the row and column index sets $T(I)$ and $T(J)$ the block cluster tree $T(I \times J)$ is constructed. The admissibility condition

$$\min\{diam(X_\tau), diam(Y_\sigma)\} \leq \eta\, dist(X_\tau, Y_\sigma) \tag{6}$$

with the admissibility factor $0 \leq \eta \leq 1$ states whether a matrix block corresponding to the cluster pairs $\tau \subset T(I)$ and $\sigma \subset T(J)$ is appropriate for a low-rank approximation or not. Here X_τ and Y_σ denote the minimal bounding boxes which include the supports of the test and trial functions of the index sets τ and σ. With eq. (6) the son operation for the block cluster tree $T(I \times J)$ is defined by

$$S_{I\times J}(\tau \times \sigma) = \begin{cases} \emptyset, & if \quad \tau \times \sigma \ fulfills\ (6)\ or\ S_I(\tau) = \emptyset\ or\ S_J(\sigma) = \emptyset \\ S_I(\tau) \times S_J(\sigma), & else. \end{cases} \tag{7}$$

The block cluster tree $T(I \times J)$ is constructed recursively by applying eq. (7) to the roots $\tau = I \subset T(I)$ and $\sigma = J \subset T(J)$ of the corresponding cluster trees. According to eq. (7) for non-admissible blocks 4 sons are generated on every level of the block cluster tree. In order to generate almost quadratic matrix blocks only clusters τ and σ on the same level are considered. The leaves of the block cluster tree define an admissible matrix partition P

$$P := \mathcal{L}(T_{I\times J}) \tag{8}$$

of the system matrix which ensures the efficiency of the H-Matrix approximation. The non-admissible blocks of P will be treated the usual way and stored as dense matrix blocks whereas the admissible blocks will be approximated by a low-rank representation (5). In order to efficiently determine low-rank representations of admissible blocks the adaptive cross approximation (ACA) is used. The ACA algorithm finds the rank k of the low-rank representation A_k for a given accuracy ε such that

$$\|A - A_k\|_F < \varepsilon \|A\|_F \tag{9}$$

is fulfilled. Since the stopping criterion is given by the condition

$$\|u_{k+1}\|_2 \|v_{k+1}\|_2 \leq \frac{\varepsilon(1-\eta)}{1+\varepsilon} \|A_k\|_F, \tag{10}$$

the rank of the approximation can be find adaptively. Compared to the FMM, the H-Matrix approach builds approximations of the system matrices K and V of equation (4) and not only of their matrix-vector products. Furthermore, based on the H-Matrix approach an arithmetic can be defined that facilitates operations such as the matrix-vector product, matrix-matrix product and matrix-inversion to be performed with quasi linear complexity. Especially, the hierarchical LU-Decomposition (HLU) of low accuracy can be used as a very efficient preconditioner, which reduces the spectral radius of the system matrix and accelerates the iterative solution process, see [3].

3. Symmetric surfaces

Besides the fast BEM algorithms of the FMM and H-Matrix compression techniques the numerical costs can be further reduced if the boundary surface is symmetric. If the surface is cylindrical, it can be subdivided into m rotationally symmetric parts. For symmetric surfaces with one symmetry plane m is equal to 2. The BEM system matrix for symmetric boundary surfaces will have a block circulant structure

$$A = \begin{pmatrix} A_1 & A_2 & \cdots & A_m \\ A_m & A_1 & \cdots & A_{m-1} \\ A_{m-1} & A_m & \cdots & A_{m-2} \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_3 & \cdots & A_1 \end{pmatrix}, \tag{11}$$

which is a special kind of Toeplitz matrix. As a result the computational time to set up the coefficient matrix is reduced by a factor of $1/m$ times the original computational time since only the blocks $A_1, \ldots, A_m$ have to be computed. Block circulant matrices, like eq. (11), have the property that they are diagonalizable by a Fourier matrix and hence the memory costs are reduced by the same factor of $1/m$ times the original memory requirements even though a direct solution scheme is applied.

By applying the H-Matrix compression to the matrix blocks $A_1, \ldots, A_m$ the computational time to set up the coefficient matrix can be further reduced. Since the matrix-vector product is very cheap compared to the diagonalizing process with the Fourier matrix, it is more favourable to use an iterative solver with an appropriate preconditioner. Taking into account that only the blocks $A_1, \ldots, A_m$ of the system matrices (4) are compressed as H-Matrices a preconditioner based on a hierarchical LU-factorization of A is not available unless the memory reduction is given up and the blocks are copied. An appropriate preconditioner which meets this requirements will be introduced in the next section. Due to the fact that a matrix-vector product of A is equivalent to m^2 matrix-vector products with the blocks $A_1, \ldots, A_m$ (m for each block) the H-Matrix technique is more advantageous than the FMM.

Symmetric or rotationally symmetric boundary surfaces occur only in a special kind of problems, hence, the direct solution strategy based on the discrete Fourier transform is limited to these cases. However, more often it is observable that boundary surfaces can be separated into (rotationally) symmetric and non-symmetric parts. The general block structure of the system matrix for a surface with 2 symmetric and 1 non-symmetric part is given by

$$A=\begin{pmatrix} D_{11} & \begin{matrix} \\ A_1 & A_2 & \cdots & A_m \\ A_m & A_1 & \cdots & A_{m-1} \\ A_{m-1} & A_m & \cdots & A_{m-2} \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_3 & \cdots & A_1 \end{matrix} D_{12} & D_{13} \\ D_{21} & & D_{23} \\ D_{31} & D_{32} & \begin{matrix} B_1 & B_2 & \cdots & B_n \\ B_n & B_1 & \cdots & B_{n-1} \\ B_{n-1} & B_n & \cdots & B_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ B_2 & B_3 & \cdots & B_1 \end{matrix} \end{pmatrix}. \tag{12}$$

Matrix blocks related to non-symmetric parts are denoted by D whereas A and B correspond to the symmetric circulant matrix blocks. In addition to the reduction of the circulant matrix blocks A and B as outlined before, the non-symmetric coupling blocks D will be treated as single H-Matrices. Due to the possible rectangular structure of some blocks, e.g. D_{12} and D_{21}, the related dofs have to be reordered and subdivided in order to generate well balanced cluster trees to ensure the efficient H-Matrix representation. However, the efficiency of this matrix reduction highly depends on the ratio of the block sizes of the symmetric and the non-symmetric parts. The less non-symmetric dofs exist, the better the advantage of circulant matrix blocks will be. Based on the aforementioned iterative solution procedure a problem with a mixed system matrix (12) can be treated the same way by exploiting the H-Matrix compression technique. The efficiency of the proposed mixed matrix reduction will be illustrated in the numerical examples section.

4. Regularization and preconditioning of the BIE

The exterior problem with Neumann boundary conditions is defined by

$$\begin{aligned} &\Delta p(\boldsymbol{x}) + k^2 p = 0 \quad \boldsymbol{x} \in \Omega \\ &\frac{\partial p(\boldsymbol{x})}{\partial n(\boldsymbol{x})} = f(\boldsymbol{x}) \quad \boldsymbol{x} \in \partial\Omega \\ &\frac{\partial p}{\partial r} - ikp = 0 \;\; r \to \infty. \end{aligned} \tag{13}$$

The corresponding discrete system derived from the boundary integral equation is given by equation (4). It is known that this system is not solvable for all frequencies due to the occurrence of irregular frequencies, see [5]. However, there are several possibilities to overcome this non uniqueness problem. Determining overdetermination points [10] or using the dual surface method [11] leads to uniquely solvable problems but they reveal problems at higher frequencies. The former suffers from the additional point placement and the latter shows difficulties with complex geometries. The most used methods to regularize the BIE are combined field integral equations. A distinction is made here according to direct and indirect regularization techniques. The indirect methods combine single layer and double layer potentials so that the physical variables on the boundary are replaced by artificial ones, see [12]. Whereas the direct methods maintain the physical variables by combining the normal derivative $\frac{\partial p(\boldsymbol{y})}{\partial n(\boldsymbol{y})}$ of the BIE (3)

$$c(\boldsymbol{y})\frac{\partial p(\boldsymbol{y})}{\partial n(\boldsymbol{y})} + \int_{\partial\Omega} \frac{\partial G(\boldsymbol{x},\boldsymbol{y})}{\partial n(\boldsymbol{y})\partial n(\boldsymbol{x})} p(\boldsymbol{x})\partial\Omega(\boldsymbol{x}) = i\rho\omega \int_{\partial\Omega} \frac{\partial G(\boldsymbol{x},\boldsymbol{y})}{\partial n(\boldsymbol{y})} v_n(\boldsymbol{x})\partial\Omega(\boldsymbol{x}) \tag{14}$$

and the BIE (3). Applying the collocation scheme to (14) will lead to the discrete system of equations

$$-Dp = i\rho\omega\left(\frac{1}{2}I - K'\right)v_n. \tag{15}$$

The most popular direct combined field integral equation is the Burton-Miller formulation [5] with a complex coupling parameter α

$$\left(\frac{1}{2}I + K - \boldsymbol{\alpha}D\right)p = i\rho\omega\left(V - \boldsymbol{\alpha}\left(\frac{1}{2}I - K'\right)\right)v_n. \tag{16}$$

If $\Im(\alpha) \neq 0$ then the problem is uniquely solvable for all real frequencies. A good choice of α is $\alpha = {}^{i}/_{k}$ [6]. However derived from the Calderón identities an optimal choice for α would be the Neumann-to-Dirichlet (NtD) map, since

$$\left(\frac{1}{2}I + K - \boldsymbol{NtD}D\right)p = Ip. \tag{17}$$

Choosing the NtD operator for α rather than a complex valued parameter for combining the integral equations improves the eigenvalue clustering of the system matrix and regularizes the problem at the same time. Hence, the NtD coupling operator is a good preconditioner of eq. (16). However, the complex valued coupling parameter $\alpha = {}^{i}/_{k}$ can be understood as the local application of the Sommerfield radiation condition

$$\begin{gathered}\frac{\partial p}{\partial r} - ikp = 0 \quad r \to \infty \\ \frac{\partial p}{\partial r} = ikp \;\to\; p = -\frac{i}{k}\frac{\partial p}{\partial r} \;\to\; NtD \approx -\frac{i}{k}\end{gathered} \tag{18}$$

on the surface $\partial\Omega$ and thus as simple approximation of the NtD operator.
A more accurate approximation of the NtD map is given by the On-Surface Radiation Condition (OSRC) method [8]. The idea of the OSRC is to find a local surface approximation of the non-local NtD operator which is derived as

$$NtD \approx \frac{1}{ik}\left(1 + \frac{\Delta_\Gamma}{k_\varepsilon^2}\right)^{-1/2}, \tag{19}$$

where Δ_Γ denotes the surface Laplace-Beltrami operator [8]. Discretization of the Laplace-Beltrami operator Δ_Γ leads to a sparse matrix with a small bandwidth. In order to prevent occurrence of singularities, a complex wavenumber $k_\varepsilon = k(1 + i\epsilon)$ is used with a damping parameter ϵ, which depends on the spatial dimension of the object. The complex root will be approximated by means of a Padé approximation of size N_p

$$\sqrt{1 + X} = C_0 + \sum_{j=1}^{N_p} \frac{A_j X}{1 + B_j X} \qquad 3 \leq N_p \leq 8, \tag{20}$$

where the factors A_j, B_j and C_0 are defined according to [8]. For the iterative solution process of eq. (16) with the GMRES method only the application of a matrix-vector product with the NtD operator (19) is required. The application of the NtD approximation (19) for a matrix-vector product requires the solution of $N_p + 1$ systems of equations. Since the resulting linear systems have the same sparsity pattern as the discretized Laplace-Beltrami operator Δ_Γ and thus a small bandwidth, the linear systems can efficiently be solved by a sparse LU-factorization. The application of the OSRC method for regularization and preconditioning for the FMM is presented in [7].

5. Numerical examples

The efficiency of the proposed method will be illustrated by means of 2 numerical examples. For this purpose a scattering problem of a submarine will be considered for an incident plane wave. The model of the submarine almost coincides with the model from the Benchmark target strength simulation (BeTTSi) workshop II. Its surface is about 60m large and discretized with at least 10 discontinuous and constant elements per wavelength, which results in 96,334 dofs for a frequency of 1kHz. The discretized surface of the submarine is symmetric with respect to the xz-plane. For the fluid parameters the properties of water with a speed of sound of $c = 1500\,m/s$ and a density of $\rho = 1000\,kg/m^3$ are assumed. All calculations are performed on a 8 core workstation with 2 quad core Intel Xeon X5647 CPUs with a clock speed of 2.93GHz each and 96GB of shared RAM. Both examples are performed with the same H-Matrix parameter set. The admissibility constant is $\eta = 0.9$ and the relative accuracy for the ACA approximation is set to $\epsilon = 10^{-5}$. Furthermore, the incident plane wave source is located in the xy-plane with an angle of incidence of 45° (w.r.t. the x-axis). The resulting system of equations will be solved by the GMRES method and a stopping criterion of 10^{-6}. For the OSRC approximation of the NtD-map a Padé expansion of size $N_p = 4$ is used. Furthermore, the frequency domain between 100Hz and 1kHz will be considered with a frequency resolution of $\Delta f = 100$Hz.

5.1 Case 1: Symmetric surface

The first case considers the scattering of a symmetric surface. Figure 1 depicts the full and the symmetric model of the submarine. According to equation (11) with $m = 2$ only half of the

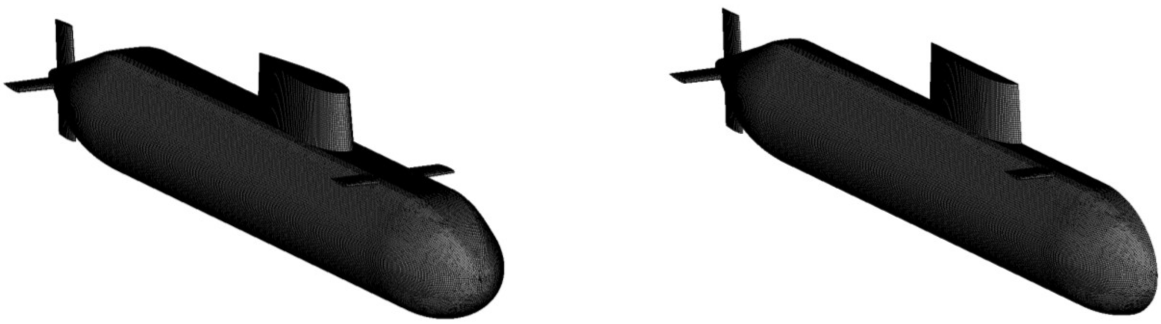

Figure 1 – Full and symmetric model of the submarine

system matrix has to be setup in the symmetric case. As illustrated in Figure 2, the effort to setup the double layer matrix are halved compared to the full H-Matrix compression. This reduction is well

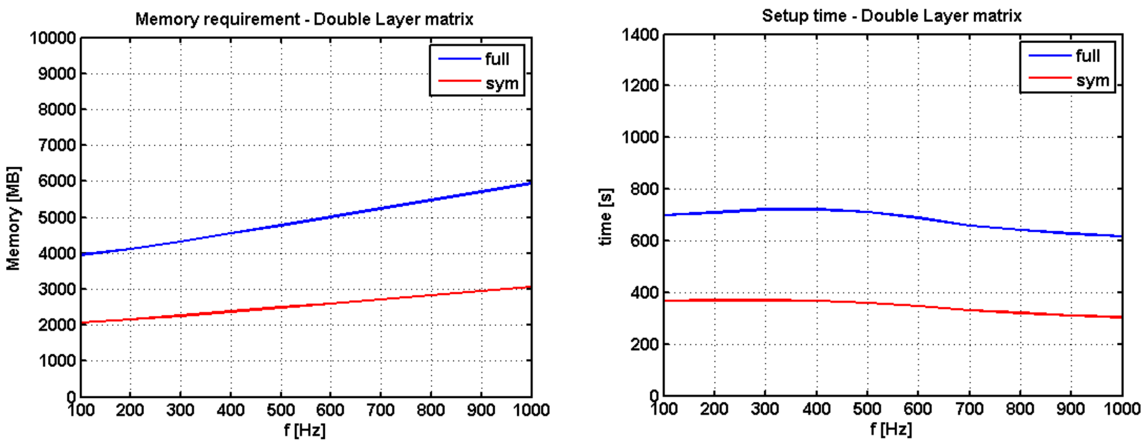

Figure 2 – Memory requirement and setup time for the double layer matrix

observable for the memory requirements as well as the setup time for the double layer matrix. For the hypingersingular matrix the same reduction effort can be observed, see Figure 3. Again the memory requirements as well as the setup time are reduced by 50%.

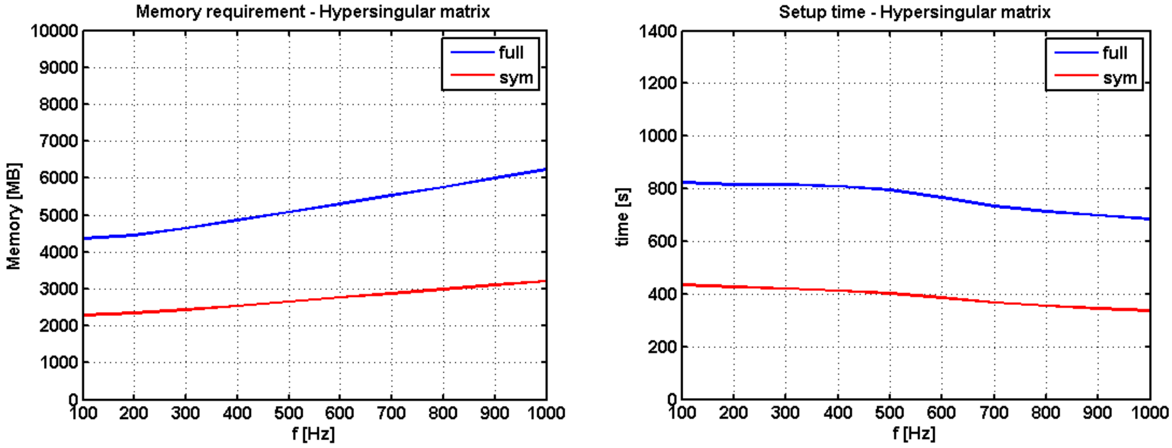

Figure 3 – Memory requirement and setup time for the hypersingular matrix

The optimal factors for the symmetric case with $m = 2$ for reduction in memory and time is 50% times the effort for the full H-Matrix approximation. Figure 4 confirms that the achieved reduction is almost obtained.

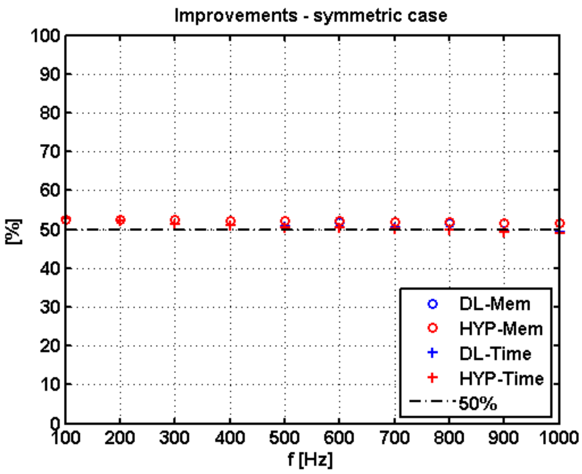

Figure 4 – Improvement factors

The comparison of the direct combined field integral equation regularization method proposed by Burton Miller for the the coupling operators $\alpha = i/k$ (BM) and $\alpha \approx NtD$ (OSRC) reveals the good properties of the OSRC NtD approximation. Figure 5 shows the superior behaviour of the OSRC preconditioner w.r.t. the number of GMRES iterations. Up to the frequency of 1kHz the number of

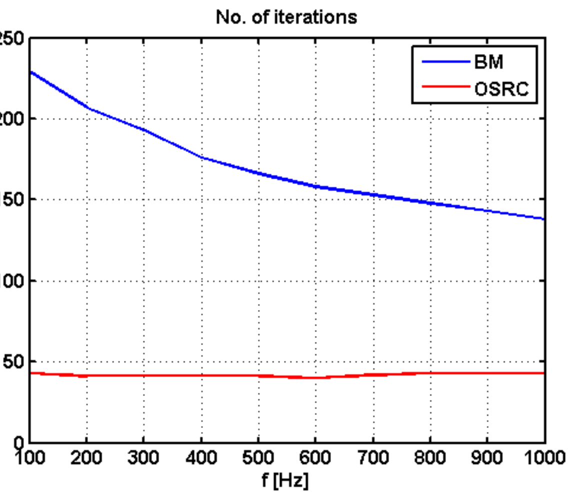

Figure 5 – No. of iterations for the symmetric problem

iterations required to solve the problem is almost constant between 40 and 43 iterations for the OSRC method. Whereas the classical approach starts with 229 iterations at 100Hz and continuously reduces to 138 at 1kHz. The resulting distribution of the total pressure over the surface of the submarine is illustrated by Figure 6. An acoustic shadow is oberseveable on the opposite side of incidence as expected.

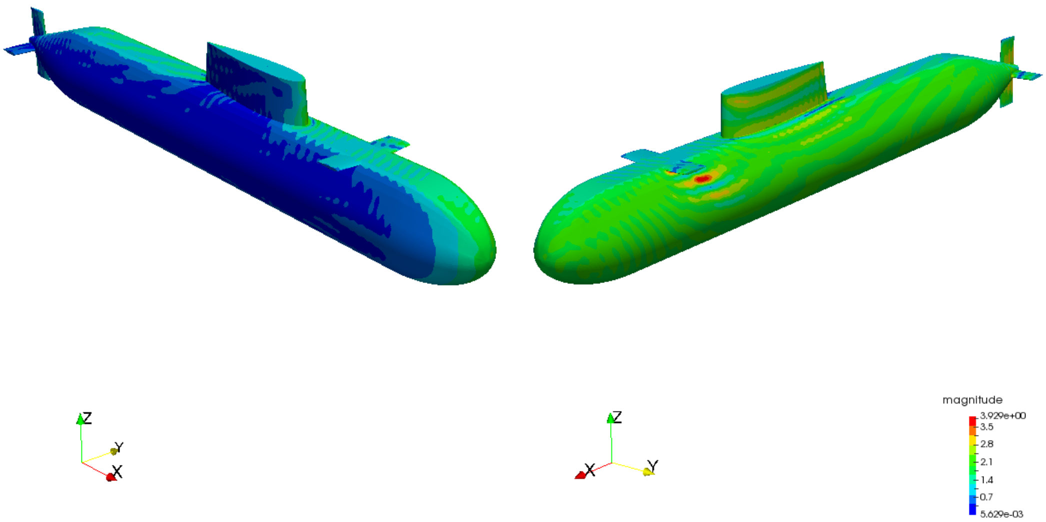

Figure 6 – Total pressure at 1kHz for the symmetric problem

5.2 Case 2: Partly symmetric surface

In order to illustrate the reduction potential for partly symmetric structures corresponding to eq. (12), the symmetric model is extended by a non-symmetric object (cyan), see Figure 7.

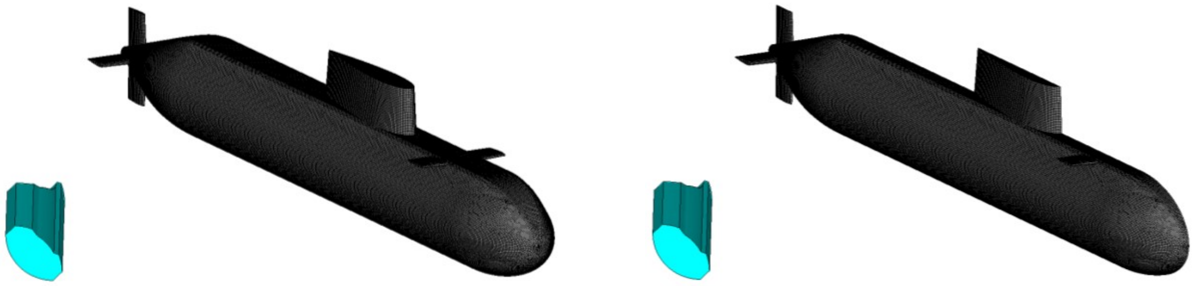

Figure 7 – Full and partly symmetric model of the submarine

Since there is solely one circulant matrix block A with $m = 2$, only the coupling blocks D_{11}, D_{12} and D_{21} are non zero. The resulting model have 115,045 dofs of which 96,334 correspond to the symmetric part and 18.711 are associated to the obstacle. In comparison to the full symmetric model, the reduction potential for the partly symmetric case is limited to the circulant matrix block A in (12). Figure 8 shows the assembling costs for the double layer matrix. It is obvious that the approximation time scales not optimal wheras the memory requirements are as expected.

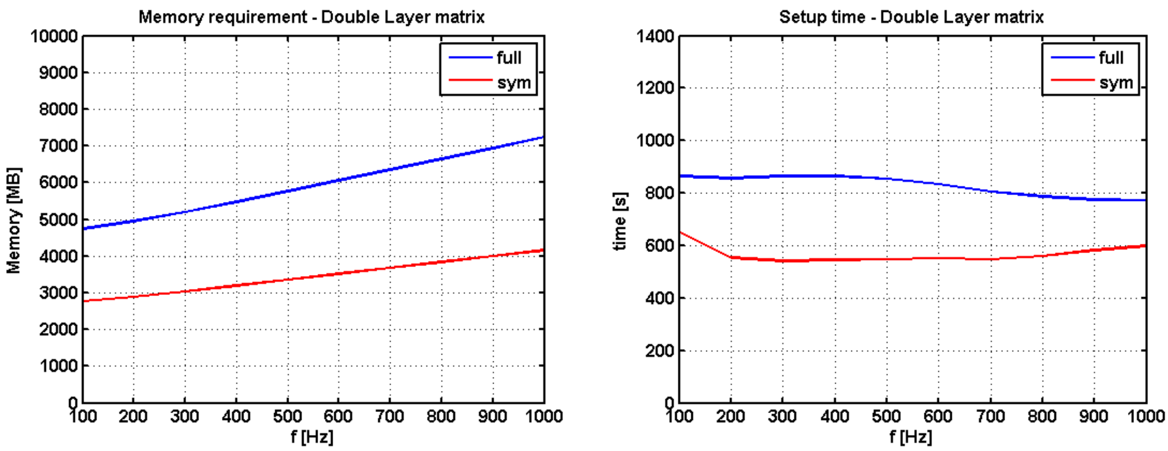

Figure 8 – Memory requirement and setup time for the double layer matrix

The improved memory reduction and the non-optimal setup time can be observed for the hypersingular matrix as well, see Figure 9.

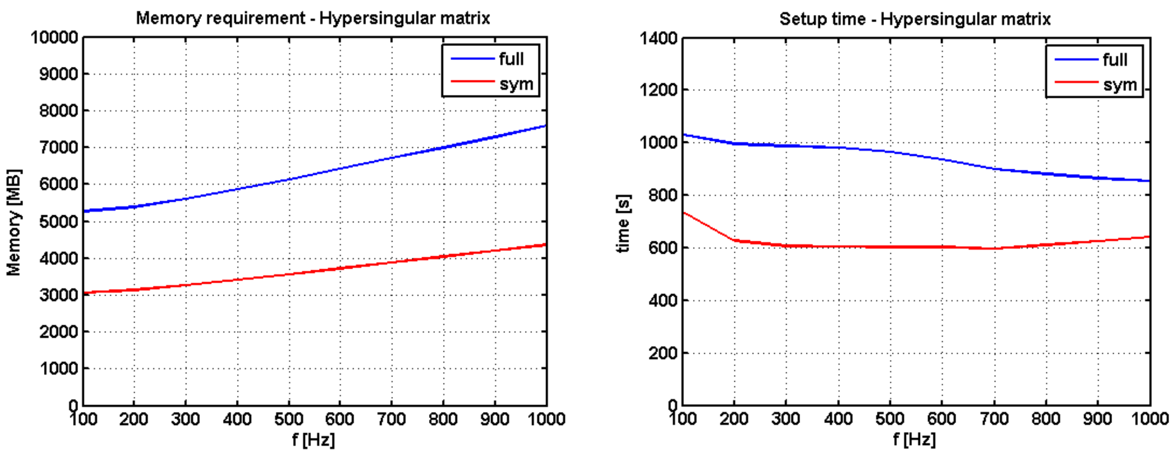

Figure 9 – Memory requirement and setup time for the hypersingular matrix

Since the potential to reduce the effort is restricted to the circulent matrix block A in eq. (12), an optimal reduction seems to be 65% of the full H-Matrix. The reduction of the memory requirement to at least 59% of the full H-Matrix is 6% better than the predicted 65%, cf. Figure 10. As the memory costs for the symmetric part remain the same, the coupling blocks will be treated as separate H-Matrices which results in a better block decompostion compared to the full H-Matrix compression and thus leads to a slightly higher compression rate. Unfortunately, the treatment of the coupling blocks is only implemented serial, hence, the setup time scales not optimally over all frequencies, which is also illustrated by figure 10.

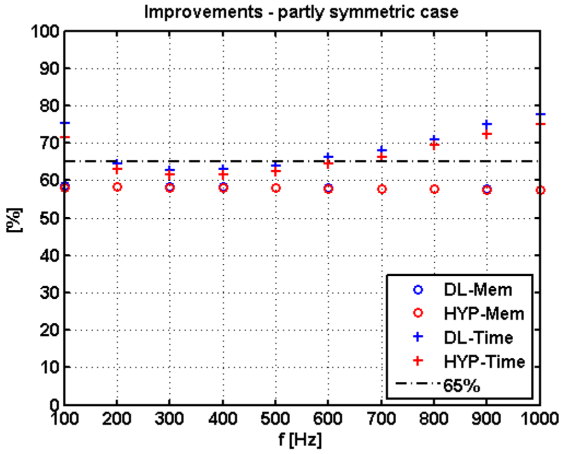

Figure 10 – Improvement factors

Also for the partly symmetric scattering case the OSRC regularization $\alpha \approx NtD$ outperforms the classical Burton Miller approach with $\alpha = i/k$ regarding the number of GMRES solver iterations. The OSRC regularization leads to an almost constant number of iterations (40 to 43) for the considered frequencies whereas the classical Burton Miller approach starts with 234 at 100Hz and decrease to 138 at 1kHz.

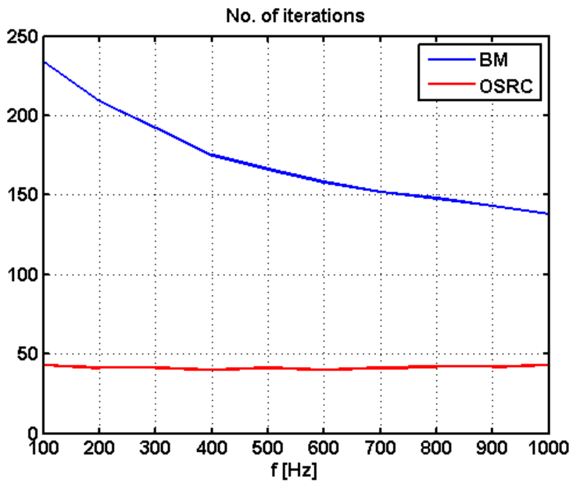

Figure 11 – Improvement factors and no. of iterations for the partly symmetric problem

Compared to the symmetric case, the influence of the obstacle on the pressure distribution is only marginally at 1kHz, see figure 12.

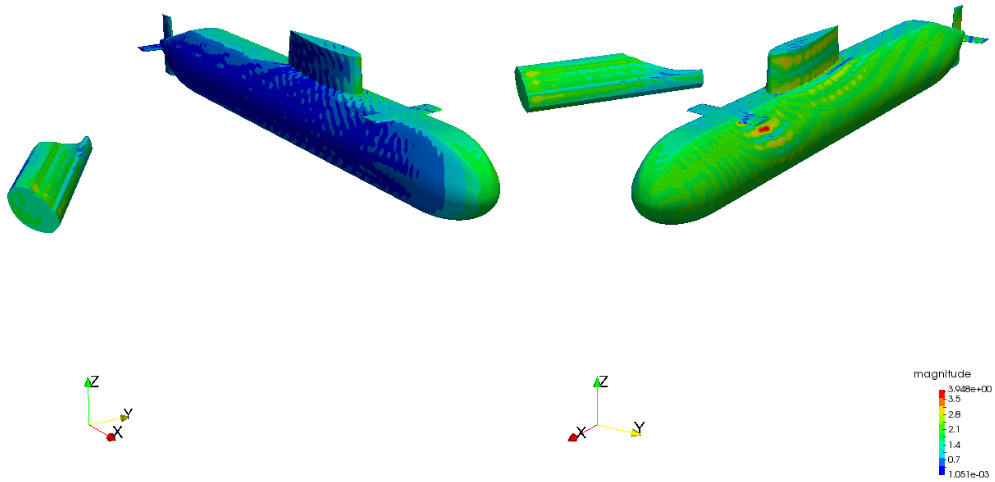

Figure 12 – Total pressure at 1kHz for the partly symmetric problem

6. CONCLUSIONS

It is demonstrated that exploiting partly symmetric boundary surfaces results in a more efficient BEM regarding computational time and memory requirements. By taking advantage of block circulant matrix parts in combination with the H-Matrix compression of matrix blocks the overall costs to set up the system matrix can significantly be reduced. For the presented symmetric problem the optimal reduction of 50% could be achieved for memory requirements as well as setup time. In case of the partly symmetric problem the achieved compression rate is even better than predicted due to a different matrix partition of the coupling blocks.

Especially, the combination with the analytic OSRC preconditioner makes this approach very efficient when applied to symmetric submarine models since a preconditioner deduced from the full H-Matrix by a low precision HLU is not necessary anymore to achieve fast convergence for an iterative solver. With the proposed method for symmetric and partly symmetric boundary surfaces larger (w.r.t. to the no. of dofs) acoustic radiation problems can be solved with the same hardware resources.

REFERENCES

1. von Estorff, O., Boundary Elements in acoustics- Advances and Applications, WIT Press, Southampton, 2000
2. Wu, T.W., Boundary Element Acoustics: Fundamentals and Computers Codes, WIT Press, 2000
3. Bebendorf, M., Hierarchical Matrices: A Means to Efficiently Solve Elliptic Boundary Value Problems, Volume 63 of Lecture Notes in Computational Science and Engineering (LNCSE), Springer-Verlag, 2008
4. Hackbusch, W., Hierarchische Matrizen: Algorithmen und Analysis, Springer-Verlag, 2009
5. Burton A. J., Miller G. F., The application of integral equation methods to the numerical solution of some exterior boundary value problems, Proc. Roy. Sot. London Ser. A 323, 1971
6. Kress R., Minimizing the condition number of boundary integral operators in acoustics and electromagnetic scattering, The Quarterly Journal of Mechanics and Applied Mathematics, 38(2), 1985
7. Darbas, M., Darrigrand, E., Lafranche Y., Combining Analytic Preconditioner and Fast Multipole Method for the 3-D Helmholtz Equation, Journal of Computational Physics Vol. 236, 2013
8. Antoine X., Darbas M., Alternative integral equations for the iterative solution of acoustic scattering problems, The Quarterly Journal of Mechanics and Applied Mathematics, 58(1), 2005
9. Tsuboi H., Sakurai A., A simplification of boundary element model with rotational symmetry in elctromagnetic field analysis, IEEE Transaction on magnetics Vol. 26, No. 5, 1990
10. Schenck, H. A., Improved Integral Formulation for Acoustic Radiation Problems, Journal of the Acoustical Society of America, 07/1968, Vol. 44, Issue 1, 1968

11. Mohsen, A., Piscoya R., Ochmann M., The Application of the Dual Surface Method to Treat the Nonuniqueness in Solving Acoustic Exterior Problems, Acta Acustica united with Acustica, Vol. 97, 2011
12. Brakhage, H., Werner P., Über das Dirichletsche Aussenraumproblem für die Helmholtzsche Schwingungsgleichung, Archiv der Mathematik, 16(1), 1965