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Isogeometric Boundary Element Method for Acoustics

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ABSTRACT

The Boundary Element Method (BEM) is an efficient numerical approach for solving an acoustical problem in the frequency domain. Due to the transformation of the differential equation onto the surface, a two-dimensional description is sufficient for the solution of a three-dimensional problem. The geometries are usually developed within CAD software that are able to represent curved shapes exactly. In the classical procedure these representations are discretized by elements based on Lagrange polynomials, in most cases by an automesh tool. This can lead to either a poor quality or manual improvement consumes a lot of time. The goal of the Isogeometric BEM is to directly incorporate the geometry description of the CAD tools, which are based on Non-Uniform Rational B-Splines (NURBS). On one hand the required mesh is easier to carry over to the BEM tool, since the underlying shape description is the same. On the other hand the surface is represented exactly in contrast to the approximation by the Lagrange polynomials. In this contribution the procedure of the new geometry description is depicted and numerical examples show the correct implementation, but also reveal the new challenges.

Keywords: 75.5 Boundary element methods (BEM)

1. Introduction

An acoustical problem in the frequency domain is based on the Helmholtz equation

$$\nabla^2 p = -k^2 p, \quad (1)$$

which depends on the acoustical pressure p an the wavenumber $k = \frac{\omega}{c}$. This partial differential equation can be transformed from the domain onto the surface, leading to the conventional boundary integral equation (CBIE)

$$c(\mathbf{x})p(\mathbf{x}) = \int_\Gamma G(\mathbf{x},\mathbf{y})\frac{\partial p(\mathbf{y})}{\partial n_y} ds_y - \int_\Gamma p(\mathbf{y})\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n_y} ds_y \quad (2)$$

This equation expresses the sound pressure p(x) at the field point x depending on the sound pressure p(y) and the acoustical flux $q = \frac{\partial p(\mathbf{y})}{\partial n_y}$ at the source point y. The boundary factor is $c(x) = 1/2$ for a smooth surface. The equation holds for interior as well as for exterior problem, since the Sommerfeld radiation condition is automatically fulfilled. For three-dimensional problems the fundamental solution reads

$$G(\mathbf{x},\mathbf{y}) = \frac{e^{\mathrm{ik}|\mathbf{r}|}}{4\pi|\mathbf{r}|}. \quad (3)$$

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A drawback of the formulation is the occurrence of the spurious eigenfrequencies. To heal this problem different methods can be applied, like the Burton Miller formulation [1]. The formulation requires the additional hypersingular boundary integral equation (HBIE)

$$c(\mathbf{x})q(\mathbf{x}) = \int_{\Gamma} [\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n_x} q(\mathbf{y}) - \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial n_x n_y} p(\mathbf{y})] \mathrm{d}\Gamma \quad (4)$$

that depends on the derivative with respect to the normal vector n_x at the collocation point. The HBIE holds for the same boundary values, but has different spurious eigenfrequencies. Hence, a linear combination with the CBIE leads to a unique solution in the whole frequency regime. The coupling factor should be $\alpha = \frac{-i}{k}$ [2] and the combined equation reads

$$(CBIE) + \alpha(HBIE) = 0 \quad (5)$$

A major problem of the HBIE is the numerical integration of the hypersingular kernel ($\frac{1}{r^3}$). Special integration routines are necessary to achieve a good accuracy.

The boundary integral equation has to be discretized in order to achieve a numerical solution. Therefore, the continuous boundary values $f(\xi)$ have to be approximated by sampling points f_m and their corresponding ansatz functions $\phi_m(\xi)$

$$f(\xi) = \sum_{m=1}^{M} \phi_m(\xi) f_m. \quad (6)$$

Usually, Lagrange polynomials of different orders are incorporated in the Boundary Element Method. For the geometry the same procedure holds to approximate the surface. Using the ansatz functions of the boundary values also for the geometry leads to the so called isoparametric concept. In isogeometric analyses the CAD description that is based on NURBS, is applied to the geometry. But generally, there is no requirement to use the same functions for the boundary values and geometry. This circumstance is used in this contribution: the geometry is discretized by NURBS and the boundary values are approximated by Lagrangian polynomials of arbitrary order.

The discretized equation can be integrated numerically, e.g. with Gaussian integration. Special integration routines are required for the singularities in the fundamental solution and its derivatives.

The system of equations is derived by the collocation method, where each collocation point influences every other point and vice versa. This leads to a fully populated system matrix A and a corresponding right hand side b

$$Ax = b. \quad (7)$$

The solution of this system of equation is either achieved by direct solution methods or iterative solvers, like GMRES, for further information of the solution process see [3]. Due to fully populated system matrix, at least a quadratic complexity arises.

2. Geometry Description based on NURBS

In this contribution the geometry for the BEM is discretized by NURBS. These ansatz functions allow an exact reproduction of curved shapes and are the common standard in CAD software. The surfaces are built from two one-dimensional NURBS curves in each of the intrinsic element coordinates. The procedure is described in detail in [4] and can be summarized as follows. In contrast to the isoparametric concept patches are introduced as another level above the elements. The functions are now defined on these patches and different forms of the ansatz functions within each element can occur. The distinct functions are based on B-Splines, knot vectors and weights defined on the patches.

2.1 Knot vector

The knot vector is a one dimensional vector in parametric space that consists of integers. These integers define the elements and their multiplicity describe the number of sampling points in each element of a patch. Unequally spaced integers in the vector lead to the term non-uniform.

2.2 B-Spline basis

The basis functions are recursively defined B-Splines

$$B_{i,0}(\xi) = 1: \qquad \xi_i \leq \xi < \xi_{i+1} 0: \tag{8}$$

else

$$B_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} B_{i,p-1}(\xi) \quad - \frac{\xi - \xi_{i+p+1}}{\xi_{i+p+1} - \xi_{i+1}} B_{i+1,p-1}(\xi)$$

with $\frac{0}{0} \equiv 1$. The order is p, the index of the function is i and the coordinate in the parameter space is called ξ and the corresponding sampling points are ξ_i.

2.3 Weights

The knot vector and the B-Spline basis functions allow a description of the so called B-Spline curves. Connecting the sampling points or control points in global coordinates linearly defines the control polygon and the multiplication with the B-Splines depicts the desired geometry. But some forms, e.g. a circle, are not constructible with this proceed. Using the methodology of projective transformation leads to new possibilities, as geometries can be represented by a projection from $R^d + 1$ to R^d, e.g. a piecewise quadratic function in R^3 can be transformed to a circle in R^2. This procedure is incorporated by the so called weights that are connected to each sampling point.

2.4 Surfaces

For the BEM, two-dimensional surfaces are required. These are built by multiplication of two one-dimensional NURBS curves. Hence, the knot vector, the B-Spline basis and the weights in each natural direction span the surface. The numerical integration is performed by Gaussian integration as in the conventional methods. The Jacobian has to be adopted according the new shape functions. Thus, in contrast to linear geometry ansatz functions a varying normal vector on the element can occur. In figure 1 and 2 a NURBS and Lagrange geometry are depicted.

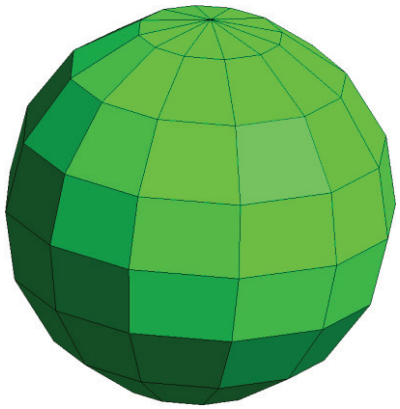

Figure 1: Lagrange discretization

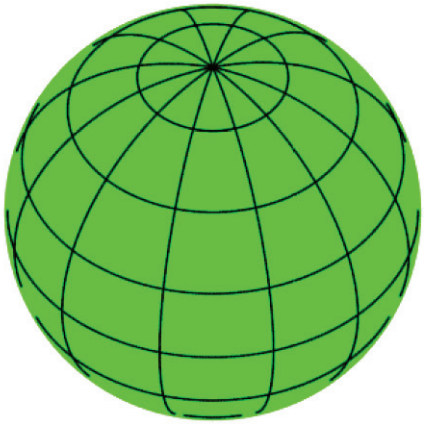

Figure 2: NURBS discretization

3. BEM based on NURBS

In the proposed BEM, the boundary values are approximated by Lagrange polynomials of arbitrary order [5] and the geometry is incorporated by NURBS. This formulation is the beginning for an isogeometric analysis that would also use the NURBS ansatz functions for the boundary values.

3.1 Discontinuous Elements

In the formulation, discontinuous elements are used, where the collocation points are connected to only one element. Hence, the normal vector is uniquely defined at any collocation point and the surface is necessarily smooth. The placement of the collocation within the elements plays a significant role for the accuracy [6]. The collocation points are set to the zeros of the Gauss-Legendre polynomials, see figure 3.

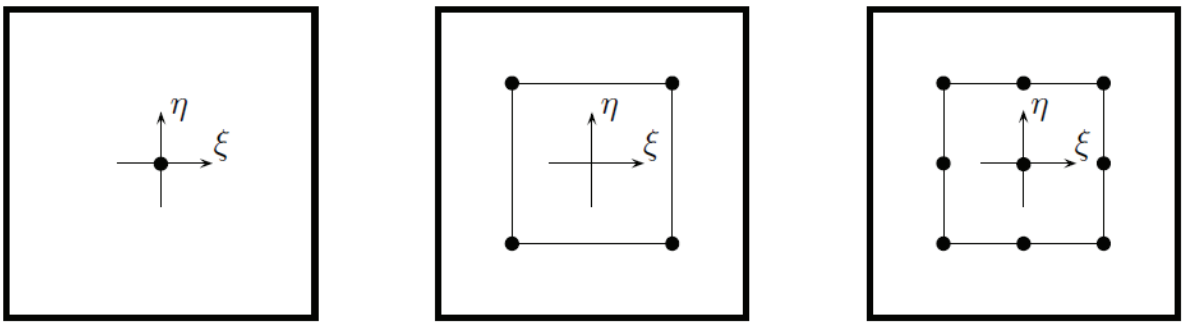

Figure 3: Discontinuous elements (constant, linear and quadratic)

In each natural direction a one-dimensional ansatz function is used. These functions for the boundary values based on polynomials of arbitrary order read

$$l_i(\xi) = \prod_{j=0, j\neq i}^{p} \frac{\xi - \xi_j}{\xi_i - \xi_j}. \quad (9)$$

3.2 Bezier extraction

The geometry is based on NURBS and the description follows from the previous chapter for two-dimensional NURBS surfaces. A drawback is the recursive formula for the B-Splines, whereas each element could have its own shape, since the functions are defined on the patches. A healing to this large numerical effort is the Bezier extraction [7]. The multiplicity of the knots in the knot vector is raised until all elements within the patches are described by the same size of knots. This leads to the

same functions, expressed as Bernstein polynomials, on each element and a decrease of the numerical effort. Although the number of sampling points is increased, the shape of the geometry stays the same.

3.3 Hypersingular Integration

One crucial part of the BEM is the numerical integration of the singularities, especially the hypersingular integration of the type $\frac{1}{r^3}$. Finding a good methodology to achieve a reasonable accuracy is an actual research topic for some decades. In the context of a Lagrangian geometry description in conjunction with a constant ansatz function for the boundary value a lot of possibilities are available. A further distinction between the collocation method and the Galerkin method has to be made. In the context of the NURBS description in conjunction with higher order elements, only a few possibilities are realized. In [7] a regularization of the whole boundary integral equation is used, but at the cost of a high numerical complexity, because an integration over complete surface has to be taken into account for each hypersingular integration. In this contribution Guiggiani's methodology [8] is newly applied to a NURBS geometry description. The method is based on subtraction of a Laurent series of the fundamental solution from the hypersingular fundamental solution. The added back term can be integrated analytically. The following steps have to be done. At first the integral has to be transformed from the intrinsic coordinates to polar coordinates around the singularity, which reduces the singularity by one due to the Jacobian of this transformation. Since the polar coordinate transformation is introduced to the natural coordinates of the element, the local radial direction ρ does not necessarily coincident with the global radial direction R. Afterwards a Laurent series expansion (a Taylor series expansion with negative exponents) around the singularity with respect to the local radial direction has to be built. Therefore, the coefficients only depend on the angular direction of the local polar coordinate transformation and the added back terms can be integrated analytically. The terms consist of the Taylor expansions of the different factors within the integral: the ansatz function, the Jacobian and the fundamental solution. Particularly, the Jacobian changes in contrast to Lagrange elements, since the geometry is defined by NURBS. The first and second derivative with respect to the local coordinates are required. The Laurent series expansion is of the same order as the hypersingular kernel and due to the subtraction a regular integral occurs, which can be integrated by Gaussian integration.

4. Numerical Examples

In this section, the proposed formulation is verified. At first the conventional BIE (CBIE) is investigated by a comparison to Lagrange geometry discretization. Thereafter, results for the Burton Miller formulation with its special integration routine is presented. The medium for all computations is air, with ρ =1.225 kg/m³ and c=340 m/s. In the study the integration order is varied and depicted by the variable g. The accuracy is measured by the Dirichlet error defined by

$$e_D = \frac{\|(\mathbf{p}_{num} - \mathbf{p}_{ref})\|}{\|\mathbf{p}_{ref}\|} \quad (10)$$

and

$$\|\mathbf{p}\| = \sqrt{\int_\Gamma |\mathbf{p}|^2 \mathrm{d}\Gamma}. \quad (11)$$

This measure gives a more precise insight, since the shape functions are introduced in contrast to a pointwise error. A Neumann boundary condition is applied based on a monopole source placed at $x_{mono} =$ [0.3 0.3 0.3]. Hence, an analytical solution is available to compute the Dirichlet error.

4.1 Conventional Boundary Integral equation

The first example is a dice with an edge length of 1m at f = 100 Hz. In figure 4 the Dirichlet error of the element order is shown. The element order defines the ansatz functions for the boundary values based on the discontinuous description. The accuracy of the Lagrange and NURBS formulation is the same, since the Lagrange elements also describe the geometry exactly. With a higher element order the Gaussian integration has to be adapted to achieve a more precise result. An integration order of g=20 leads a steadily decreasing error with an order up to p=3.

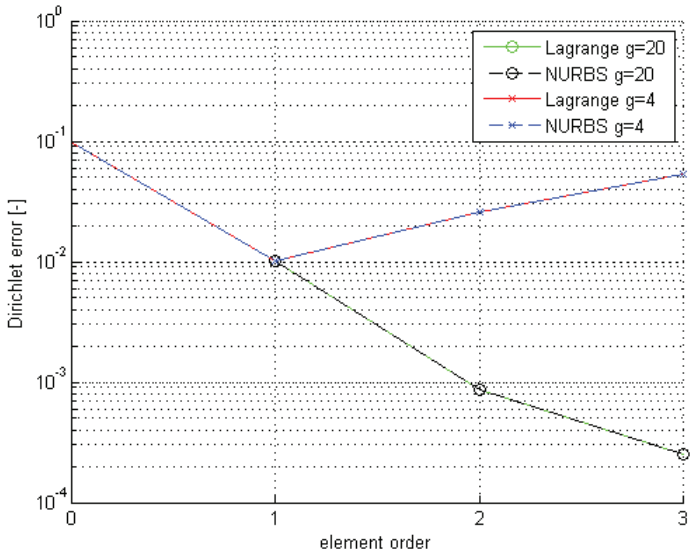

Figure 4: Dice, Dirichlet error over the element order

The next example deals with a sphere consisting of 96 elements, either based on Lagrange polynomials or NURBS. In figure 5 the Dirichlet error over the element order is depicted at frequency of f = 100 Hz. It is obvious that the NURBS discretization leads to a more accurate result than the Lagrange discretization, due to the exact geometry representation. A crucial point is at the element order p=3 that has a poor result. Probably, this is influenced by the strongly singular kernel ($1/r^2$) of the double layer potential. An increasing element order results in a high number of collocation on one element, which correlates to the number of singular integrals.

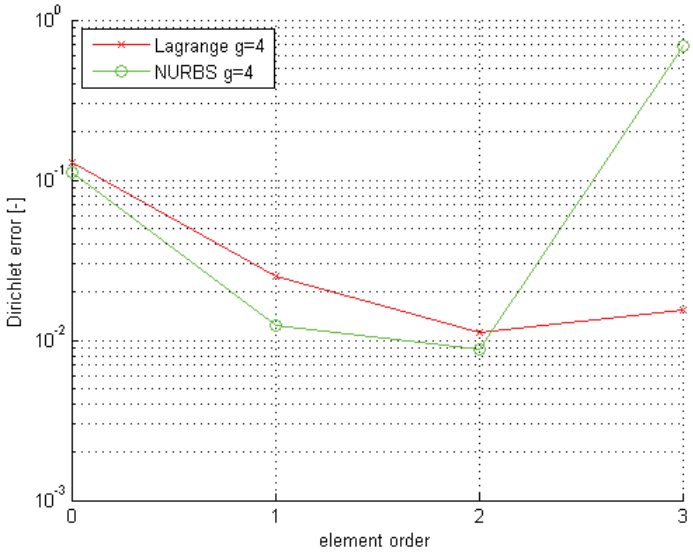

Figure 5: Sphere, Dirichlet error over the element order

The following calculations shall show the advantage of a direct combination with the CAD geometry. In figure 6 the Dirichlet error over the number of elements is presented and in all examples a quadratic element order is used at a frequency of f = 100 Hz. Hence, the degrees of freedom are 9 times higher (3 by 3 collocation points per element) than the number of elements. The usual way of today's simulation procedure is to build only one Lagrange mesh from the CAD geometry. If a finer discretization is required, e.g. for a higher frequency, the Lagrange elements are just split if no conjunction to the CAD geometry is available or the meshing process would be too complicated.

Therefore, the geometry is not approximated more precisely. This procedure is named “Mesh” in the figure. In contrast to this, if a conjunction to the CAD geometry is available and a mesh is build each time newly the geometry is approximated more accurate. This is indicated by “CAD”. In the proposed formulation an automatic link to CAD is available. This third option gives the most accurate result in this example and confirms the advantage of this procedure.

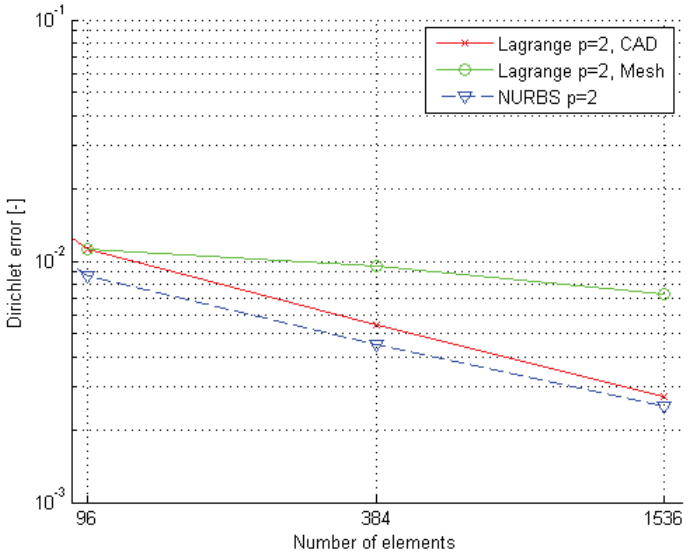

Figure 6: Sphere, Dirichlet error over the number of elements

4.2 Burton Miller formulation

The next example shows the frequency dependent behavior of the Burton Miller formulation (BM). A frequency sweep with a step of f = 2.5 Hz is applied to the CBIE, the HBIE and the combination BM. The problem is the dice from the first investigation with an edge length of 1m and 96 elements. In figure 7 the Dirichlet error over the frequency is plotted for constant and quadratic elements. The solution of the HBIE shows clearly the spurious eigenfrequencies. For the CBIE the eigenfrequencies are hardly seen, but one can be found at f = 297.5 Hz. It seems that these frequencies have such a sharp shape that a finer frequency discretization would be required to see the effects. As expected, the BM formulation shows for both element orders a unique solution.

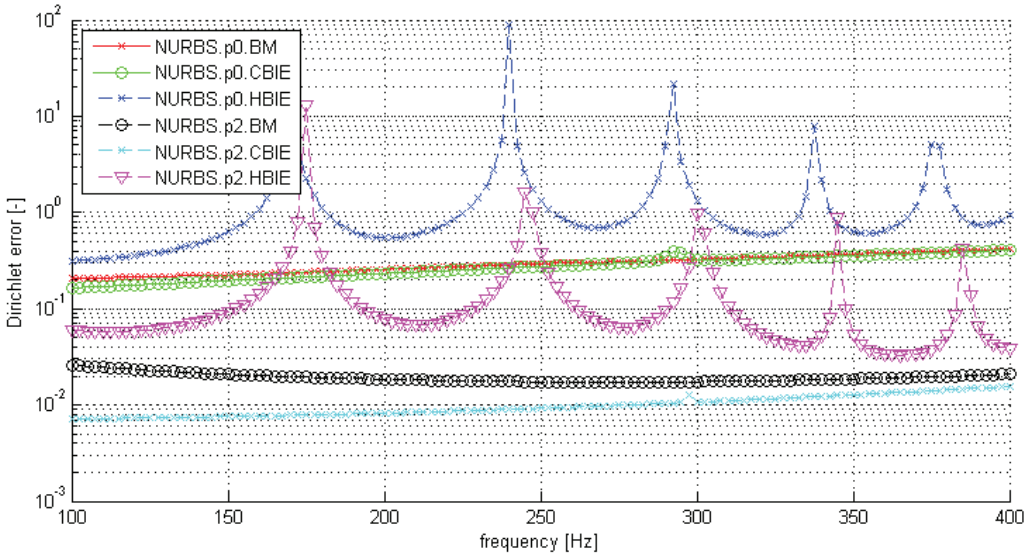

Figure 7: Dice, Dirichlet error over the frequency

The last example shows the difference between a NURBS and Lagrange discretization for the Burton Miller formulation. A sphere with a radius of r = 1m and described 384 elements is analyzed for the CBIE, the HBIE and the combined BM. In figure 8 the Dirichlet error over the frequency is depicted. Both formulations show the spurious eigenfrequencies, which are more visible for the CBIE than in the case of dice. A slight deviation of the frequency position can be observed, which is probably based on the different geometry between Lagrange and NURBS. Looking at the values for the error a marginal advantage for the NURBS in contrast to the Lagrange geometry can be seen.

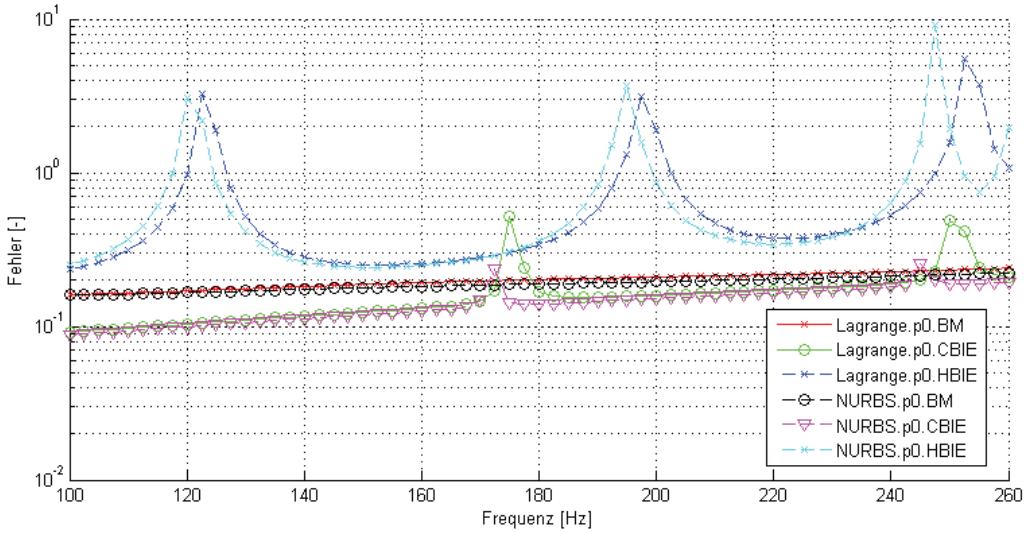

Figure 8:Sphere, Dirichlet error over the frequency

5. Conclusions

In this contribution the principles for an Isogeometric Boundary Element Method are depicted. The geometry description is based on NURBS and the ansatz functions for the boundary values are defined by Lagrange polynomials of arbitrary order on discontinuous elements. Due to the spurious eigenfrequencies, the Burton Miller formulation is incorporated and a way of handling the hypersingular integrations is presented. The results show the advantages of the NURBS formulation, especially, if refinements are necessary and the linking to the CAD geometry is automatically fulfilled.

Interesting topics for the future are the investigation of more realistic CAD geometries, special integration routines for the strongly singular integrals or nearly singular integrals, the use of different ansatz functions for the boundary values as well as the combination with fast BEM methods. The proposed formulation could be accelerated by the H-matrices or the Fast Multipole Method.

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