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Thai fiddle saw-u modeled as a Helmholtz resonator with circular membrane

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Abstract

Saw-u is a Thai low-pitched vertical fiddle, of which two strings are bowed to vibrate, via a bridge, a sheet of goatskin or cowhide stretched over a cavity with sound holes. This instrument is similar to Cambodian tro-u, but distinguished from Chinese yehu or Korean haegeum in that animal skin is used for the interface to the bridge (rather than wood in the latter two). In the current study, the unique structure of the instrument body was investigated by establishing a mathematical model where the cavity was assumed to behave as a Helmholtz resonator interacting with a circular membrane. Two coupled equations governing the motions of the membrane and that of the air mass in the sound holes were solved based on simple assumptions. The results showed that the resonance frequencies associated with the circular modes of the membrane were shifted under the influence of the Helmholtz resonator, whereas that of the resonator would remain unchanged. Moreover, it was found that an additional circular mode may be observed near the Helmholtz resonance frequency, which may significantly influence the sound quality by reinforcing the resonance. The application of the current findings to the analysis of similar structures will also be discussed.

Keywords: Saw-u, Thai fiddle, Helmholtz resonator, vibrating membrane, modal analysis

1 INTRODUCTION

Saw-u is a two-stringed, low-pitched fiddle found in Thailand (see Figure 1). When the string is bowed, the vibration is transferred, via a bridge made of a small piece of wood, to the thin animal skin (membrane), stretched over the front cut of a coconut shell (resonator) with sound holes on the other side. The saw-u is similar to Cambodian *tro-u*, but distinguished from Chinese *yehu* and Korean *haegeum* for which thin wooden sheets are used in place of animal skin.

The saw-u has been the subject of some previous studies mostly in Thailand. In the study by Malin and Meesawat (1), the transfer function of the saw-u resonator was measured and used as a filter to synthesize the sound. Punwaratorn (2) compared the sound quality of the two resonators made from alternative materials with that from the conventional (coconut shell). However, the details of the acoustic properties of the saw-u had not been discussed in these studies.

The purpose of this study is to look into the fundamental acoustic properties of the saw-u resonator. The resonator was simplified into a coupled system consisting of a membrane-cavity (or kettledrum-like system) and a Helmholtz resonator. A Newtonian equation of motion was established to analyze the resonance characteristics of the system. Similar models can be found in a few previous studies. For example, Fletcher and Thwaites (3) discussed a number of biological acoustic systems, one of which is the directional ear, consisting of a diaphragm on one side and a port on the other. The frequency and directional response of this system could be analyzed with the equivalent circuit method. Also, Christensen and Vistisen (4) presented a simple system of the guitar in an attempt to study its low frequency response. The model is composed of a plate attached to a spring (acted by the guitar top plate) and an air mass (in the guitar sound hole), both of which are under the influence of the restoring force acted by the air volume in the instrument body. A Newtonian equation of motion was used to analyze the system, and the nature of the coupling frequencies and the frequency response of the system could be accurately predicted. In the two studies mentioned in the previous text, the vibrating diaphragm or plate was

Figure 1. The front, back, and side view of a Thai saw-u.

assumed to be a rigid mass. In the current study, however, the membrane is treated as a vibrating membrane having radial and circular modes.

In section 2, the model for the saw-u resonator is presented, followed by section 3 where the implications of the model are discussed and compared with previous models. Lastly, a summary is given in section 4.

2 A SIMPLE MODEL OF THE SAW-U RESONATOR

2.1 Derivation

The saw-u resonator may be modeled as a combination of three components (see Fig. 2): A circular vibrating membrane of area $S_M = \pi a^2$, areal density σ, and surface tension T, a cavity of volume V_0, and a cylindrical tube with effective length L and cross-sectional area S_H.

Helmholtz resonator is composed of the cavity and the tube. The air in the tube is considered to be a rigid mass or an *air piston* with mass $\rho_0 S_H L$ (ρ_0: equilibrium density of air), whereas the air in the cavity acts as a spring exerting a force $-S_H \Delta P$ to the air piston. The negative sign indicates the direction opposite to the displacement of the piston. $\Delta P(t)$ is the instantaneous pressure deviation from the equilibrium pressure P_0. This model is valid for wavelength λ that satisfies the following: $\lambda \gg V_0^{1/3}$, $\lambda \gg S_H^{1/2}$, and $\lambda \gg L$ (5). Also, the cavity modes and the resistance (elastic resistance and radiation loss) are neglected in the current model.

Without external force, the motion of the air piston resembles a simple mass-spring system. For $\boldsymbol{\xi}(t)$, the instantaneous displacement of the air piston, the equation of motion is represented as $\rho_0 S_H L \ddot{\boldsymbol{\xi}} = -S_H \Delta P$, or

$$\rho_0 L \ddot{\boldsymbol{\xi}} = -\Delta P. \quad (1)$$

Assuming that the pressure inside the cavity is uniform and the membrane displacement is small, it exerts a force $\Delta P \, \mathrm{d}x \mathrm{d}z$ to an infinitesimal element of the membrane surface. When $\boldsymbol{y}(x,z,t)$ represents the instantaneous displacement of the infinitesimal elements (using Cartesian coordinate for convenience) and $\ddot{\boldsymbol{y}}$ the second deriva-

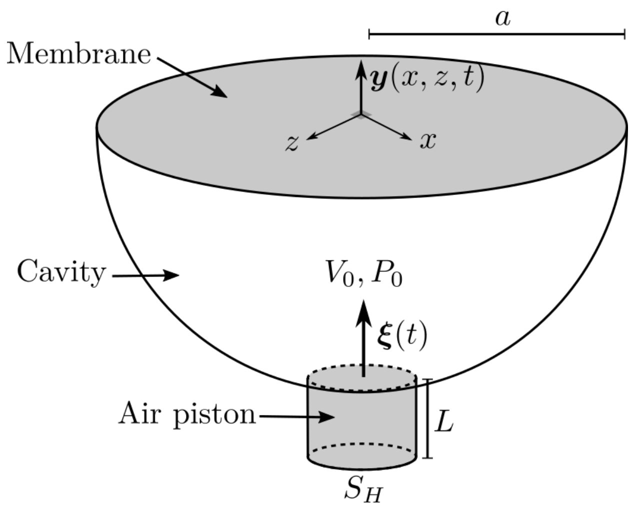

Figure 2. The schematic of the simple model of saw-u resonator for mathematical analysis.

tive in time, the equation of motion for the membrane can be described as $\sigma \ddot{\mathbf{y}} \mathrm{d}x\mathrm{d}z = T\nabla^2 \mathbf{y} \mathrm{d}x\mathrm{d}z + \Delta P \mathrm{d}x\mathrm{d}z$, or

$$\sigma \ddot{\mathbf{y}} = T\nabla^2 \mathbf{y} + \Delta P. \tag{2}$$

The pressure difference $\Delta P(t)$ results from the volume change $\Delta V(t)$ and can be approximated as

$$\Delta P \approx -\gamma \frac{P_0}{V_0} \Delta V, \tag{3}$$

where γ is the ratio of the heat capacity at constant pressure to that at constant volume of the air in the cavity. Here ΔV is attributed to the volume displacement of the air piston, $S_H \xi$ and that of the membrane, $S_M \langle \mathbf{y} \rangle$, where $\langle \mathbf{y} \rangle$ is the average displacement of the membrane which can be expressed as $\langle \mathbf{y} \rangle = S_M^{-1} \int_{S_M} \mathbf{y} \, \mathrm{d}S_M$. So, ΔV can be expressed as

$$\Delta V = S_M \langle \mathbf{y} \rangle - S_H \boldsymbol{\xi}. \tag{4}$$

Substituting Equations (3) and (4) into (1) and (2) gives

$$\ddot{\boldsymbol{\xi}} + \omega_H^2 \boldsymbol{\xi} = \frac{S_M}{S_H} \omega_H^2 \langle \mathbf{y} \rangle \tag{5}$$

$$\nabla^2 \mathbf{y} - \frac{1}{c_M^2} \ddot{\mathbf{y}} = \frac{S_H}{T} \gamma \frac{P_0}{V_0} \left(\frac{S_M}{S_H} \langle \mathbf{y} \rangle - \boldsymbol{\xi} \right), \tag{6}$$

where $\omega_H^2 = S_H \gamma P_0 / \rho_0 L V_0$ is the Helmholtz resonance frequency (assuming a rigid wall instead of a membrane) and $c_M = \sqrt{T/\sigma}$ is the speed of the transverse wave on the membrane.

2.2 Solution

Assuming harmonic solutions for both $\mathbf{y}(r,\theta,t)$ and $\boldsymbol{\xi}(t)$, i.e. $\mathbf{y}(r,\theta,t) = \Psi(r,\theta)e^{i\omega t}$ and $\boldsymbol{\xi}(t) = \Xi e^{i\omega t}$ respectively. Equations (5) and (6) can be rearranged:

$$\Xi = \frac{S_M}{S_H} \frac{\omega_H^2}{\omega_H^2 - \omega^2} \langle \Psi \rangle, \tag{7}$$

$$\nabla^2\Psi + k^2\Psi = \frac{S_H}{T}\gamma\frac{P_0}{V_0}\left(\frac{S_M}{S_H}\langle\Psi\rangle - \Xi\right). \tag{8}$$

Equation (7) indicates that Ξ and $\langle\Psi\rangle$ are in-phase for $\omega < \omega_H$, and out-of-phase for $\omega > \omega_H$.
Finally, combining Equations (7) and (8) gives the decoupled equation of motion for the membrane:

$$\nabla^2\Psi + k^2\Psi = \frac{S_M}{T}\gamma\frac{P_0}{V_0}\left(\frac{\omega^2}{\omega^2 - \omega_H^2}\right)\langle\Psi\rangle. \tag{9}$$

The homogeneous solution to this equation is Bessel function, similar to the solution of a free membrane. However, a particular solution is required for $\Psi(a)$ to be zero. Also, the only affected modes are the axisymmetric ones (modes with only circular nodal lines) because $\langle\Psi\rangle$ becomes zero for the modes with radial nodes and so the equation becomes homogeneous. From now, only the axisymmetric modes will be discussed. Given the boundary condition, the general solution can be represented as $\Psi = A[J_0(kr) - J_0(ka)]$. Equation (9) becomes

$$J_0(ka) = -\frac{S_M a^2}{T}\gamma\frac{P_0}{V_0}\left(\frac{J_2(ka)}{(ka)^2 - \omega_H^2 a^2/c_M^2}\right), \tag{10}$$

which is a conditioning equation for the resonance frequencies of the system.

3 IMPLICATIONS AND COMPARISONS

Equation (10) is similar to the conditioning equation for a membrane over a cavity or a kettledrum mentioned in Kinsler et al. (5), with $(ka)^2$ in the denominator replaced by $(ka)^2 - \omega_H^2 a^2/c_M^2$. The presence of ω_H modifies the resonance frequencies of the membrane modes. Two limiting cases can be evaluated. First, for the case where ω_H approaches zero (when the sound holes are blocked), the system is reduced to a kettledrum. When ω_H approaches infinity (when the sound holes are extremely large or the cavity vanishes), the equation is reduced to $J_0(ka) = 0$, which is the conditioning equation for the free circular membrane vibration. So, the conditioning equation is consistent with the expected behaviors of the system in the limiting cases.

To estimate the resonance frequencies of the system for ω_H in a practical range, the conditioning equation can be rearranged:

$$\omega_H = \left[\frac{S_M}{\sigma}\gamma\frac{P_0}{V_0}\left(\frac{J_2(ka)}{J_0(ka)}\right) + \frac{c_M^2}{a^2}(ka)^2\right]^{1/2}, \tag{11}$$

where the intersections between the left and right sides of the equation indicate the resonance frequencies of the system. Using Equation (11), ω_H can be represented by a horizontal line, which is equivalent to the values in y-axis. In this way, the resonance frequencies of the system (with fixed parameters; S_M, c_M, etc.) can be directly represented by the the right side of the equation as shown by the solid curves in Figure 3. Parameters used in Figure 3, chosen to resemble an actual saw-u, are listed in Table 1. In the specific case of $\omega_H = 308$ Hz, the four resonances, indicated by intersections between the ω_H line and the conditioning curves, are 162, 329, 415 and 638 Hz.

In Figure 3, three axisymmetric modes are shown for free membrane (177, 406 and 637 Hz) and kettledrum (211, 410 and 638 Hz). However, the current model predicts the presence of an additional mode. At low ω_H, the additional resonance (lowest one) is slightly below ω_H, while other resonance frequencies are slightly above the corresponding kettledrum resonance frequencies: The higher the frequency, the smaller the shift. As ω_H rises, the lowest resonance shifts toward the first axisymmetric resonance frequency of a free membrane and the second resonance shifts in parallel with, and slightly above, ω_H. This shifting nature is consistent with the findings in Christensen and Vistisen (4). In addition, this shift can be observed at about 400 and 640 Hz, where the resonance frequencies of free membrane and kettledrum are found. Furthermore, no resonance frequency exists between the free membrane and kettledrum modes.

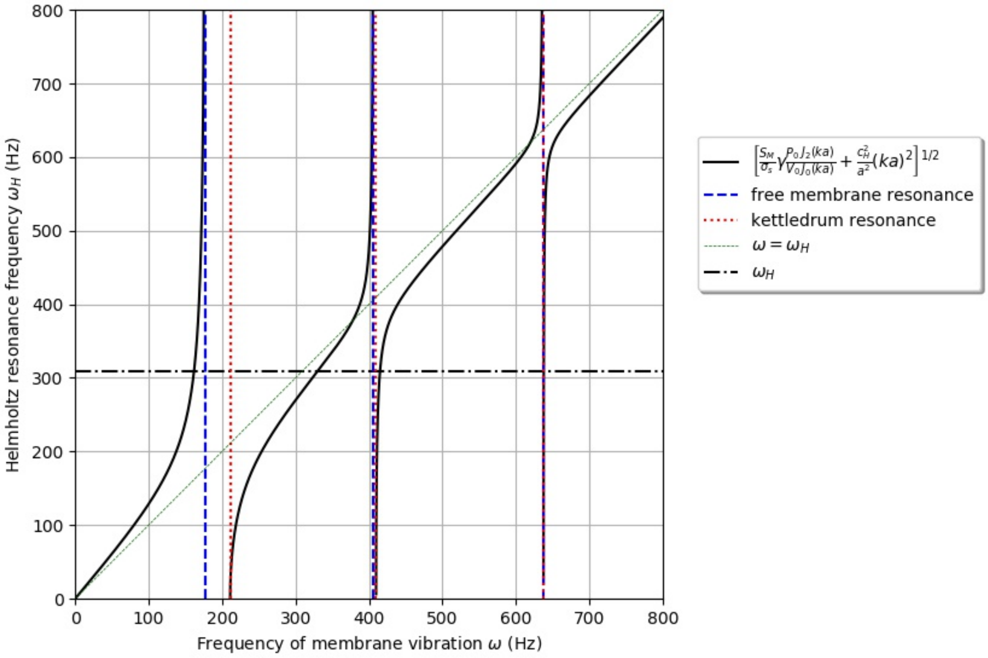

Figure 3. The right side of Equation (11) (black curves). Green thin dotted line indicates $\omega = \omega_H$. Vertical lines are the resonances of the fixed-rimed membrane in vacuum (free membrane; blue dashed lines) and those of the membrane stretched over the closed cavity (kettledrum; red dotted lines). Black dotted-dashed line is ω_H at 308 Hz.

Table 1. Parameters used in Figure 3.

Parameters	symbols	values	units
Membrane radius	a	6.0	cm
Membrane surface area	S_M	113.1	cm^2
Membrane density	σ	1.3	$\mathrm{kg\,m^{-2}}$
Wave speed on membrane	c_M	27.74	$\mathrm{m\,s^{-1}}$
Equilibrium air pressure	P_0	1013.25	hPa
Cavity volume	V_0	1600	cm^3
Helmholtz resonance frequency	$\omega_H/2\pi$	308	Hz

Comparisons are made between the current model and the two previous ones as shown in Table 2. Only the first two resonance frequencies are compared because the previous models can only estimate the lowest two resonance frequencies. All predictions indicate that the lowest resonance frequency is lower than that of the first kettledrum mode and the second is higher than ω_H.

The limitation of the current model is that the frequency range is limited by the assumptions regarding the rigid-body motion of the air piston, as is the case for the model by Christensen and Vistisen. Since the saw-u is a low pitched instrument, however, the model should still be applicable.

Table 2. Comparison of the first two resonance frequencies between different models.

Resonance	Models		
	Current paper	Fletcher and Thwaites	Christensen and Vistisen
Lower	162 Hz	161 Hz	163 Hz
Upper	329 Hz	339 Hz	336 Hz

4 SUMMARY

The saw-u resonator was simplified into a combination of a Helmholtz resonator with circular membrane. By solving a set of Newtonian equations of motion, the conditioning equation for the resonance frequencies of the saw-u resonator was established. The current model suggested that an additional mode of vibration is present along with the kettledrum and free membrane modes. These membrane mode frequencies shift as the Helmholtz resonance frequency varies. The lowest two resonance frequencies were found to agree with those predicted by two previous models. Measurements of a saw-u resonator are to be carried out to verify the findings of the current study.

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