BEM-model to simulate the vibrations in a tunnel in layered orthotropic media

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Introduction

Due to an increase in heavy traffic and the construction of rail roads near or in residential areas, models for the prediction of vibrations in soil become more and more important. We are presenting a 3-dimensional BEM-model of a tunnel going through a horizontally layered orthotropic material. The goal of the simulation is to calculate the deformations at the tunnel walls, the tunnel base and at the soil surface caused by a load at the tunnel base.

Until now, there is no known analytical form of the fundamental solution for this kind of problems. Since, however, it is needed for a BEM-approach, a numerical approximation for this function has to be calculated. This will be done in the Fourier domain, which has two advantages: the original 3-dimensional problem can be decoupled into smaller, independent 2-dimensional problems and there is no need to integrate over a singularity of the fundamental solution, because it vanishes in the Fourier domain.

The Fourier back transform and the solution of the boundary integral equation (BIE) are exchanged, because the special form of the fundamental solution allows an analytical solution of some of the integrals occuring in the BIE. The back-transform from the Fourier domain is then done numerically.

Fundamental Solution

The soil is modelled as a horizontally layered half-space. Each layer has its own set of parameters: the Young's moduli E_i, the shear moduli G_{ij}, the Poisson ratios ν_{ij} ($i, j \in \{x, y, z\}$), density ρ and thickness d. Underneath the last regular layer an additional half-space layer is added with appropriate boundary conditions to prevent unwanted ("unphysical") reflections.

In a first step the deformations and stresses for the layered half-space *without* the tunnel, caused by loads in different depths and directions, will be calculated (here only a sketch is given - for more details refer to [1, 2]). Based on these results an approximation for the fundamental solution can be constructed, which, in turn, will be used to set up and finally solve the BIE.

The problem of wave propagation without body forces inside a *single* (homogeneous) layer, using the Fourier-transform and Hooke's law $\boldsymbol{\sigma} = \boldsymbol{F}^{-1}\boldsymbol{D}\boldsymbol{u}$, where $\boldsymbol{\sigma}$ are the stresses, $\boldsymbol{u}$ the deformations, $\boldsymbol{F}$ is the stiffness matrix for an orthotropic media and $\boldsymbol{D}$ is a matrix representing the differentiation operators, can be stated as:

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{0} , \tag{1}$$

where $\boldsymbol{K}$ is a symmetric 3×3 matrix depending on the material parameters, the angular frequency ω and the angular wavenumbers k_x, k_y and k_z (for more details see [1]).

The system of linear equations (1) has nontrivial solutions iff $\boldsymbol{K}$ is singular. The determinant of $\boldsymbol{K}$ as a function of k_z is a polynomial of degree 6, therefore, for every pair (k_x, k_y), every layer and every frequency, we have at most 6 different values $k_z^{[j]}$ $(j = 1, ..., 6)$ for which the determinant is 0, thus rendering $\boldsymbol{K}$ singular. In the following we will call the $k_z^{[j]}$ *franz-values* (FRanz Ain't a Normal kZ-value) and the corresponding vectors $\boldsymbol{\Psi}^{[j]}(k_x, k_y) \in \ker\boldsymbol{K}(k_x, k_y, k_z^{[j]})$ *franz-vectors* of the system.

The solution to (1) for *one* $k_z^{[j]}$ then is

$$\hat{\boldsymbol{u}}^{[j]}(k_x, k_y) = a_j \boldsymbol{\Psi}^{[j]}(k_x, k_y) , \tag{2}$$

where $a_j \in \mathbb{C}$. This solution can be expanded to the entire k_z-domain, yielding

$$\hat{\boldsymbol{u}}^{[j]}(k_x, k_y, k_z) = a_j \boldsymbol{\Psi}^{[j]}(k_x, k_y)\delta(k_z - k_z^{[j]}) , \tag{3}$$

where δ denotes the Dirac-Delta functional. The reason for this is that the solution to equation (1) is $\boldsymbol{0}$ if $k_z \neq k_z^{[j]}$.

By adding the six $\hat{\boldsymbol{u}}^{[j]}$, we get the general solution to (1), which after a Fourier back transform with respect to k_z is:

$$\hat{\boldsymbol{u}}(k_x, k_y, z) = \frac{1}{2\pi}\sum_{j=1}^{6} a_j \boldsymbol{\Psi}^{[j]}(k_x, k_y) e^{\mathrm{i}k_z^{[j]}z} . \tag{4}$$

Since $\hat{\boldsymbol{\sigma}} = \boldsymbol{F}^{-1}\hat{\boldsymbol{D}}\hat{\boldsymbol{u}}$ (the transformed version of Hooke's law), the stresses can be written as

$$\hat{\boldsymbol{\sigma}}(k_x, k_y, z) = \frac{1}{2\pi}\sum_{j=1}^{6} a_j \boldsymbol{F}^{-1}\hat{\boldsymbol{D}}^{[j]}\boldsymbol{\Psi}^{[j]} e^{\mathrm{i}k_z^{[j]}z}, \tag{5}$$

where $\hat{\boldsymbol{D}}^{[j]}$ is a matrix representing the derivative operators in the Fourier domain.

The a_j in (4) and (5) have to be set such that $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{\sigma}}$ satisfy the prescribed conditions at the layer boundaries. For the last layer (the non-reflecting half space) the weights need to be set to satisfy the Sommerfeld radiation condition, i.e. such that $\hat{\boldsymbol{u}} \to 0$ (and consequently $\hat{\boldsymbol{\sigma}} \to 0$) as $z \to +\infty$.

Since we set the body forces to zero, we can only apply forces at the layer boundaries. For forces applied at a depth z_p inside the layer (which is necessary for the setup of the BIE), it has to be split into two parts, thus introducing two *virtual* layers.

Here one could perform a Fourier back transform in the y-direction, set up and solve the BIE in the (k_x, y, z)-domain and still have the advantages of a 2.5D-problem [2]. We, however, stay in the (k_x, k_y, z)-domain.

Boundary Integral Formulation

In order to use the boundary element method (BEM), it is in general necessary to have an explicit form of the fundamental solution. Looking at the problem from the engineering point of view (cf. [3]), the fundamental solution can be interpreted as the reaction of the system at an evaluation point $\boldsymbol{x}$ to an infinite point load applied at the load point $\boldsymbol{\xi}$.

With this in mind, it is clear that the algorithm described above can be used to construct an approximation for this function. In order to use this approach, we have to apply a load in the depth of every load point (i.e. everywhere, where we want to know the deformations), and evaluate the deformations and stresses at every evaluation point (i.e. on the tunnel-boundary, over which we will later integrate).

Our mesh consists of straight lines (see Figure 1) with one load and one evaluation point per element, which coincide at the midpoint of the element. That means, that if we want to know the deformations and stresses on the tunnel walls and the tunnel bottom, we have to add an addional layer for the midpoint of every BEM-element (if a symmetric mesh is used, only half the number of layers is necessary).

As the main boundary integral equation the body-force free Somigliana's identity (cf. [3]) is used, which reads (in simplified Einstein notation):

$$u_i(\boldsymbol{\xi}) = \int_\Gamma u^*_{ij}(\boldsymbol{\xi}, \boldsymbol{x}) t_j(\boldsymbol{x}) d\boldsymbol{x} - \int_\Gamma t^*_{ij}(\boldsymbol{\xi}, \boldsymbol{x}) u_j(\boldsymbol{x}) d\boldsymbol{x} \ , \quad (6)$$

where Γ is the tunnel boundary, u^*_{ij} are the fundamental deformations in the x-, y-, and z-directions ($j = 1, 2, 3$) at the evaluation point $\boldsymbol{x} = (x, y, z)$ caused by a load applied at $\boldsymbol{\xi} = (\xi, \eta, \zeta)$ in the x-, y-, and z-directions ($i = 1, 2, 3$). Note that $u^*_{ij}(\boldsymbol{\xi}, \boldsymbol{x})$ is shift invariant in x- und y- direction, i.e. $u^*_{ij}(\boldsymbol{\xi}, \boldsymbol{x}) = u^*_{ij}(x-\xi, y-\eta, \zeta; 0, 0, z)$. The fundamental stresses t^*_{ij} are defined as follows:

$$t^*_{ij}(\boldsymbol{\xi}, \boldsymbol{x}) := \sum_k^3 \sigma^*_{ijk}(\boldsymbol{\xi}, \boldsymbol{x}) n_k(\boldsymbol{x}) \ , \quad (7)$$

where $\sigma^*_{ijk}(\boldsymbol{\xi}, \boldsymbol{x})$ is the jk^{th}-component of the stress tensor at $\boldsymbol{x}$ caused by a unit load in the i-direction at the point $\boldsymbol{\xi}$, and $n_k(\boldsymbol{x})$ is the k^{th} component of the normal vector pointing outwards at $\boldsymbol{x}$, thus in our model $\boldsymbol{n} = (0, -\sin(\alpha), -\cos(\alpha))^T$ (cf. Figure 1).

As a boundary condition, we set $t_j(\boldsymbol{x}) = 0$ ($j = 1, 2$) at every element and $t_3(\boldsymbol{x}) = 0$ at every element, except for those where the load is applied on the tunnel bottom (cf. Figure 1).

As was shown in [2] it is possible to consider the BIE seperately for every k_x, thus reducing the integrations in equation (6) to simple line integrals over the tunnels cross section Γ_0 in the (y, z)-plane:

$$\begin{aligned} 0 = U_i(k_x, \eta, \zeta) - \\ - \int_{\Gamma_0} T^*_{ij}(k_x, \eta - y, \zeta; 0, 0, z) U_j(k_x, y, z) d(y, z) + \\ + \int_{\Gamma_0} U^*_{ij}(k_x, \eta - y, \zeta; 0, 0, z) T_j(k_x, y, z) d(y, z) \ , \quad (8) \end{aligned}$$

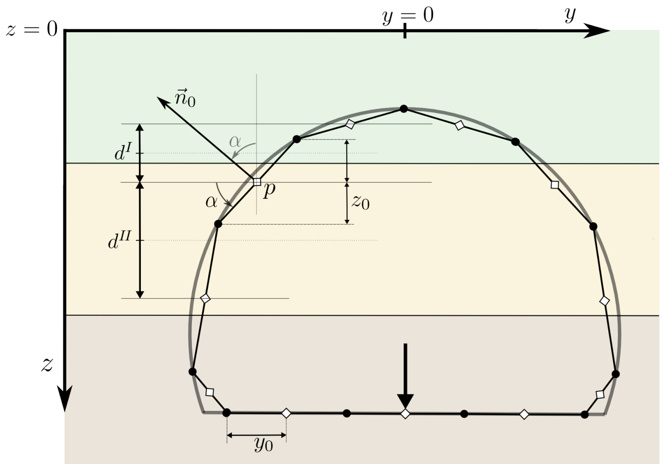

Figure 1: Scheme of the tunnel cross section.

Evaluating the Integrals

Because of the nature of the fundamental solution, the integrals in equation (6) become singular as $\boldsymbol{\xi} \to \boldsymbol{x}$, which is, from a numerical point of view, undesirable. In addition to that, the σ_{ij} are non-decaying as $k_y \to \pm\infty$ if load and evaluation point have the same depth, which is a serious problem for a numerical Fourier back transform (cf. equations (9) and (11)).

These two problems can be avoided by doing all computations in the Fourier domain (i.e. using the fundamental solution in the form that was discussed above and in [1]), but this domain, however, is unbounded. Consequently, this procedure only is advantagous if the occuring integrands tend to 0 fast enough as $k_y \to \pm\infty$, so that a numerical integration over an unbounded domain is still feasible. That this really is the case can be seen in Figure 2.

First we take a look at the second integral in equation (8): $\int_{\Gamma_0} U^*_{ij} \ T_j \ d(y, z)$. Since the load is only applied at the tunnel base and in z-direction, the integrand can be non-zero only for $j = 3$ and on horizontal elements in a constant depth z_p. After splitting the integral over Γ_0 into a sum of integrals over the elements of the discretisation Γ_p, we set the origin (in y-direction) of the local coordinate system to the midpoint of the element y_p. Because we are only looking at horizontal elements, the integration over Γ_p is reduced to $c\int_{-y_0}^{y_0} U_{i3}^{*}dy$, where c is the strength of the applied force and $2y_0$ is the length of the element.

Using the Fourier transform and exchanging the order of integration we have

$$\int_{-y_{0p}}^{y_{0p}} U_{i3}^{*}(\zeta; y + y_p - \eta, z_p)\ dy =$$
$$\int_{-\infty}^{\infty} e^{-ik_y(\eta-y_p)}\hat{U}_{i3}^{*}(\zeta; k_y, z_p) \int_{-y_{0p}}^{y_{0p}} e^{-ik_y y}dydk_y =$$
$$\int_{-\infty}^{\infty} e^{-ik_y(\eta-y_p)}\hat{U}_{i3}^{*}(\zeta; k_y, z_p)\mathrm{sink}(k_y, y_{0p})dk_y\ , \quad (9)$$

where $\mathrm{sink}(k_y, y_0) := 2\frac{\sin(k_y y_0)}{k_y}$.

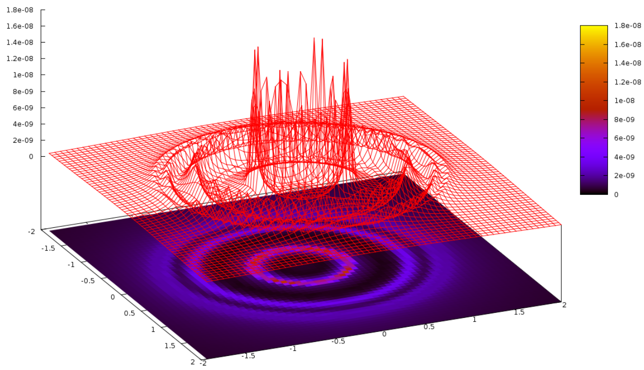

Figure 2: Absolute value of the integrand in equation (9) for $(k_x, k_y) \in [-2, 2] \times [-2, 2]$ for our test problem. The load and evaluation depths are the same.

Even in the case where the load and the evaluation elements are the same, the deformations $U_{i3}(k_y)$ tend to 0 as $k_y \to \pm\infty$ and the sink-function adds additional damping. Thus, the integrand decays fast enough, as can be seen in Figure (2) and therefore the above integral can be calculated using numerical methods.

For the first integral in equation (8) $\int_{\Gamma_0} T_{ij}^{*}\ U_j\ d(y, z)$ things are not as simple, because the elements of the discretisation Γ_p that have to be considered here can have arbitrary orientation and we propose the following approach: we use constant shape functions for the deformations, so the U_j can be taken outside the integral. Since the elements are straight lines, they can be represented by $y = g_p z + y_p,\ z \in [-z_0, z_0]$, where g_p is the slope of the element and $2z_0$ is the extension of the element in z-direction. Performing the Fourier back transform and exchanging the order of integration we get

$$\int_{\Gamma_p} T_{ij}^{*}(\zeta; y - \eta, z)d(y, z) =$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}k_y(y_p-\eta)} \int_{-z_0}^{z_0} \hat{T}_{ij}^{*}(\zeta; k_y, z)e^{\mathrm{i}k_y g_p z}dzdk_y\ . \quad (10)$$

The next step is to use the knowledge about the composition of the T_{ij} (cf. equations (4) and (5)) which (again) allows us to do the integration in z-direction analytically, finally giving the following integrand:

$$e^{\mathrm{i}k_y(y_p-\eta)}\sum_{s=1}^{6} a_i^{[s]}(\zeta)\mathbf{\Phi}_j^{[s]}(l, \alpha)\mathrm{sink}(k_z^{[s]} + k_y g_p, z_0)\ , \quad (11)$$

where $\mathbf{\Phi}_j^{[s]}(l, \alpha)$ are the properly scaled stresses calculated from the the franz-vectors and equations (5) and (7). Again, the sink-function provides damping and leads to fast enough a decay for numerical integration.

The advantages over a model that performs the Fourier back transform at an earlier stage are that in the model presented here we can do the integration in z-direction analytically and, owing to the damping by the sink-function, the numerical integrations over k_y are more accurate.

Example

For a test it was assumed that the grid used is fine enough so that U_{ij}^{*} and T_{ij}^{*} can be set constant on each element.

We assumed the tunnels cross-section to be a circle with radius 3 m and the centre at 9 m below surface. 1,5 m below the centre the circle is cut by a horizontal line, thus giving the tunnel a horizontal floor. This cross-section was discretised with 101 elements with lengths from 0.10 m to 0.18 m.

The soil consist of three layers of different thickness, additionally a half space layer was added to prevent unwanted reflections. For the parameters see Table 1.

A load of $c = 0,25\ N$ was applied at $x = 0$ and $y = 0$ at the tunnel bottom pointing in the z-direction. All calculations were made for a frequency of 40 Hz.

Layer	Parameter				
	$E_x = E_z$	$G_x = G_z$	$\nu_x = \nu_z$	ρ	d
1	9.0E7+ 3.6E6i	3.4E7+ 1.4E6i	0.330	1500	7.0
2	3.1E8+ 1.2E7i	1.2E8+ 4.7E6i	0.314	1750	2.0
3	9.0E7+ 3.6E6i	3.4E7+ 1.3E6i	0.330	1500	5.0
HS	3.6E8+ 1.4E7i	1.4E8+ 5.4E6i	0.315	2000	∞

Table 1: Material parameters for the test problem.

Results

Figure 3 shows the absolute value of the deformations in z-direction at the tunnel walls, the tunnel bottom and at the surface.

Outlook

For now, our main focus lies on improving the integration over the unbounded domain. From experience we know that the integrands in (9) and (11) are rather smooth

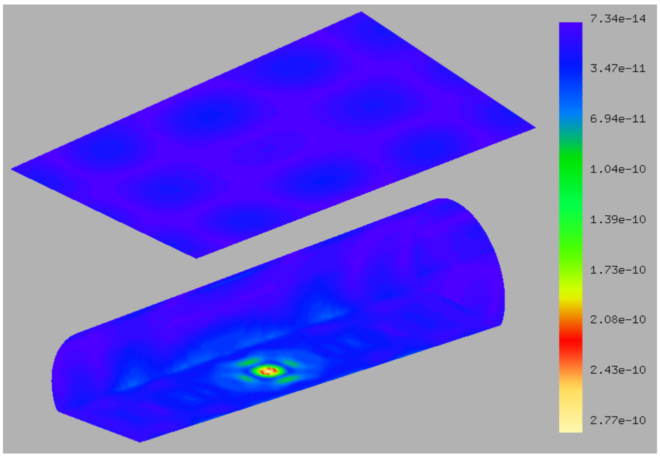

Figure 3: Absolute value of the deformations of the tunnel walls and the surface in z-direction at 40Hz.

outside a small interval around $(k_x, k_y) = (0, 0)$. Therefore it seems feasible to use different (non-equidistant) grids; e.g. a fine grid for $(k_x, k_y) \in [-3, 3] \times [-3, 3]$ and some wider grid outside this domain. In addition to that, because of the smoothness of the integrands outside $[-3, 3] \times [-3, 3]$, we suggest to approximate them with a single exponential function.

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