Simulation of Pulsed Ultrasonic Wave Propagation in Viscous Fluid exhibiting Frequency Power Law Attenuation

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Introduction

The motivation of our research is the characteristics of attenuation α found for ultrasonic wave propagation in mammalian tissue. Measurements reveal that in the frequency range used in medical ultrasonics applications (1 to 10 MHz), attenuation increases with increasing frequency. Data can be approximated with a frequency power factor

$$\alpha = \alpha_0 \cdot |\omega|^y, \tag{1}$$

where α_0 is a material dependent constant, ω the angular frequency, and y is usually a noninteger number. Duck [1] indicates, that for most soft tissue and biological fluids y lies in the range of 1.0 to 1.5, although some tissues possess a value smaller than 1.0. We aim to integrate this damping behavior including dispersion into the finite element method (FEM) code developed at the department.

Physical Model

Classical lossy time domain wave equations exist only for the restricted cases of $y = 2$, i.e. damping proportional to omega squared. In this case one often speaks of thermoviscous damping. One way of deriving the lossy wave equation exhibiting frequency power law behavior, is to start in the frequency domain and append the wavenumber $k = \frac{\omega}{c}$ by an attenuative term,

$$k(\omega) = \frac{\omega}{c(\omega)} + \jmath\alpha(\omega). \tag{2}$$

Kramers-Kronig relations applied to ultrasonics states [2], that phase velocity can be deduced form the attenuation factor by Hilbert transform. Prerequisites of the Kramers-Kronig relation to exist are square integrability and causality of the transfer function $H(\omega) = e^{jk(\omega)d}$ with d being the propagation distance. From the Hilbert transform of α one recognizes, that c is not constant but dispersion occurs. The relative dispersion reads

$$\beta'(\omega) = \frac{\omega}{c(\omega) - c_0} = \alpha_0 tan(\frac{y\pi}{2})\omega(|\omega|^{y-1} - |\omega_0|^{y-1}), \tag{3}$$

where c_0 is a reference phase velocity at frequency ω_0. It was shown[3], that under the restriction of the so called smallness approximation attenuation and dispersion can be combined to a single term. The lossy linear wave equation reads

$$\triangle p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} + \frac{2\alpha_0}{c_0 sin[(y-1)\frac{\pi}{2}]} D^{y+1} p = 0, \tag{4}$$

where D^{y+1} denotes the fractional derivative operator of order $y + 1$ in the time domain.

Numerical Implementation

We use two different algorithms to approximate the fractional derivative term in the time domain. The first algorithm is based on generalized finite differences and is mostly referred to by the names of Grünwald and Letnikov in literature. See e.g. Gorenflo [4] for a description. Luise Blank published a collocation approximation with polynomial splines [5], which we used as second algorithm. In theory the computation of fractional derivatives requires the whole history of the function in a weighted form. In contrast, derivatives of integer order depend solely on the local behavior. Both the Grünwald-Letnikov method and the spline collocation approach can be written in the form

$$D^y p_{n+1} = \sum_{k=0}^{n+1} w_k \cdot p_{n+1-k}, \tag{5}$$

where p_n is a discrete function with $p_n = 0$ for $n < 0$. We incorporated two techniques to reduce memory demands in computing the fractional derivative. At first, it can be shown for the Grünwald-Letnikov formula, that the series of weight factors $|w_{k+1}|$ is strictly decreasing from the moment on, where k becomes larger than the order of derivative y. If one assumes the amplitude values to be uniformly distributed, the more distant a function value lies in the past, the less influence has it on the value of the fractional derivative to be calculated. Starting from this fact, it was suggested to neglect addends near the starting point, i.e. in the distant past, and only calculate the sum form values of the recent past. Podlubny [6] termed that "short-memory principle". Secondly, we propose the technique of storing only every second function value. The missing function values is linearly interpolated by two stored values when computing the fractional derivative.

Discussion

We evaluated our implementation with a plane progressive wave example, where the solution can be calculated analytically in the frequency domain. In Fig. 1 one can see the difference in amplitude values between using the just presented damping model with a frequency power factor of 1.2 and classical thermoviscous damping. The value of α_0 is chosen to be equal for both models at the carrier frequency (here 1 MHz), which causes the maximum and minimum pulse amplitude to be almost the same. The relative error of the thermoviscous model is in this example 3.0 % and 0.7 %, respectively. Because the envelope of the pulse is damped differently due to its spectrum, we can find a much greater relative error

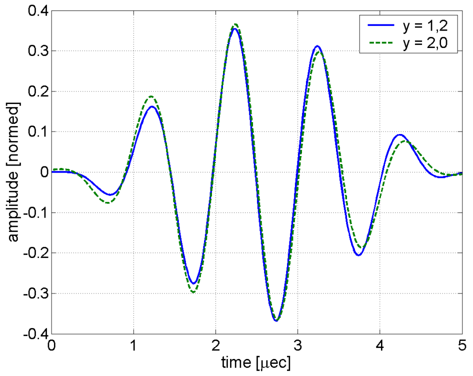

Figure 1: Ultrasonic pulse propagation after 10 cm for frequency power law and thermoviscous damping (dispersion effects are not considered).

in pulse energy of 4 %. In nonlinear wave propagation the distortion effects are expected to be even stronger because of the transfer of energy to higher harmonics, so the error becomes greater. As pointed out in the Numerical Implementation section, we follow the short-memory principle. Fig. 2 shows the influence of the number of stored past pressure values on the relative error in maximal amplitude of the pressure value resulting from FEM simulation compared to the reference values calculated in the frequency domain after a propagation distance of 5 cm. For the used grid spacing and time step size there

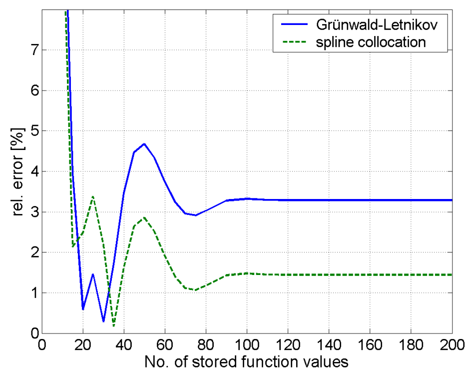

Figure 2: Influence of the number of stored function values on numerical error

would be a total number of 1910 pressure values available to compute the damped pulse. The pulse itself consists of 251 values. We can see that beyond about 90 values there is only a marginal gain in accuracy. In Fig. 3 the newly proposed strategy of interpolating function values for a further reduction in memory demands is examined for both algorithms. We can show, that linear interpolation of intermediate past function values preserves the accuracy.

Conclusion

With the presented damping model, we are able to simulate pulsed wave propagation over a distance of several centimeters with an numerical error of less than 1%. In our future work we endeavor to integrate the damping

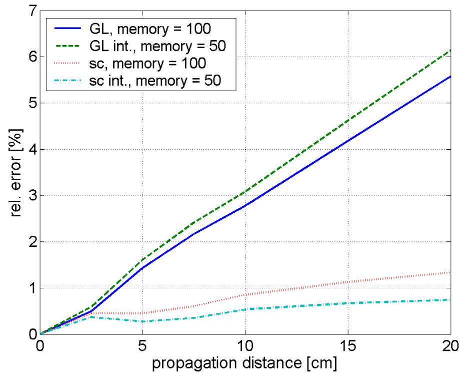

Figure 3: Comparison of the two algorithms with and without linear interpolation in relative error in maximal amplitude of intermediate past function values (GL: Grünwald-Letnikov algorithm, sc: spline collocation algorithm, int: interpolation, i.e. 50 true values and 49 interpolated ones)

model in nonlinear wave equations like Kuznetsov's equation [7]. We are planning to use higher order interpolation to further reduce the number of stored past function values and calculate heating from dissipated energy.

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