Numerical BEM/FEM formulation applied to plane wave diffraction by elastic inclusion

S. Soua[1,2], H. Bel Hadj Salah[2] , M. Ben Tahar[1], A. Dogui[2]
[1] University of Technology of Compiègne. Laboratory Roberval, Acoustic sector BP20529 60205 Compiègne, France
[2] National School of Engineer of Monastir, Laboratory of Mechanical Engineering, Avenue Ibn El Jazzar 5019 Monastir, Tunisia

Abstract :

We study the ultrasonic wave transmission and diffraction in homogeneous and isotropic medias. In this aim, we use a coupled numerical formulation by boundary and finite element methods (BEM/FEM).

The domain concerning limited inclusion is treated by finite elements, the unlimited domain is handled by boundary elements method.

The coupled formulation is validated using analytical solution. Stoneley interface mode and plane wave diffraction by cylinder are presented in this work.

Introduction

The modelling of the NDT experiments by sound waves has been the object of several works. The use of numerical methods asks for heavier means but allows raising a big number of restrictive hypotheses for analytical methods. We present here a numerical coupling method BEM/FEM applied to elastodynamic problems.

The finite element method allow modelling diverse anisotropic elastic inclusions. The boundary element method allow implicit using of radiation condition at the infinity when unlimited homogeneous elastic domain is considered.

The coupled methods differ generally, one of the others, by the choice of the integral formulation in use, or by the consideration of the conditions of continuity…

Costabel *et all* [1] presented a general mathematical concept of the use of a coupled formulation BEM / FEM for the acoustoelastic harmonic problem. They use a variational formulation in displacement form FEM, and a variational direct integral representation BEM. Writing continuity conditions for displacement and stress vectors on the coupling interface in integral form allows obtaining a non-symmetric global formulation.

Polizotto [2] provides many forms of variational approach in FEM domain based on Hu-Washizu or Hellinger-Reissner instead of the variational displacement form. In BEM domain, he uses a variational approach based on indirect integral representations. All forms are used in static case, their extension in the elastodynamic does not raise particular problems.

Han-Hou [3] presents some coupled BEM/FEM formulation applied to static problems. He uses a variational in displacement form of FEM domain, and variational BEM formulation based on direct integral representation. The continuity of the normal stress to the interface between both domains, written in two constituents among which the one obtained using the complete representation, allows him to obtain a symmetric global formulation. The extension of this approach in the elastodynamic is immediate.

We resolve the interface coupled problem in elastic wave propagation, where the BEM is based on indirect approach. Classical variational approach FEM in displacement form is used. Analytical developpements validates the numerical model.

Motion equations

In 2D and linear elastodynamic, we consider a body Ω as shown in Figure 1. $\Omega = \Omega_1 \cup \Omega_2$, with regular boundary $\Sigma = \Sigma_c \cup \Sigma_u$.

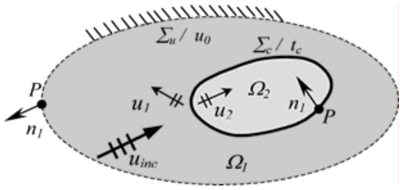

Figure 1. Geometry of the interface problem.

The governing equation for displacement vector is expressed as:

$$div(\sigma_i) + \rho_i \, \omega^2 u_i = 0, \; i = 1..2 \, , \tag{1}$$

where σ_i is the stress tensor in Ω_i:

$$\sigma_i = C_i : \varepsilon_i \quad i = 1,2 \, . \tag{2}$$

An incident field is considered in Ω_1, the total displacement is decomposed in u_{inc} and u_1:

$$u_{1T} = u_1 + u_{inc} \, . \tag{3}$$

u_1 satisfies the radiation condition at infinity.

Ω_1 and Ω_2 are coupled by the boundary Σ_c, and continuity is applied to total components of displacement and stress:

$$\begin{cases} u_{1T}(M) = u_2(M) & (a) \\ t_{1T}(M) = t_2(M) & (b) \end{cases} \quad M \in \Sigma_c \, . \tag{4}$$

Formulations

The coupling process used in this work is the following:

In the variational principal applied to Ω_2 we use continuity condition (4) b thus t_2 is replaced with the integral representation. The continuity condition for displacements and the boundary condition on Σ_u are applied to displacements in integral form for u_1.

Finally, we obtain three integral equations with three unknown variables u_2 in Ω_2, ψ on Σ_c and σ on Σ_u.

The coupling boundary Σ_c is divided into linear elements (L2). A piecewise linear variation along each boundary element is assumed for unknowns ψ and σ. The displacement u_2 in Ω_2 is interpolated with linear shape functions and (T3) element is used.

We obtain a non symmetric assembled matrix given by :

$$\begin{bmatrix} [\Xi_2] & -[\Sigma] & [T] \\ -[I] & \tfrac{1}{2}[I]-[T] & [G] \\ 0 & -[T] & [G] \end{bmatrix} \begin{Bmatrix} u_{2/\Omega_2} \\ \psi_{/\Sigma_c} \\ \sigma_{/\Sigma_2} \end{Bmatrix} = \begin{Bmatrix} -T_{inc/\Sigma_c} \\ -U_{inc/\Sigma_c} \\ U_{o/\Sigma_u} - U_{inc/\Sigma_u} \end{Bmatrix} . \tag{5}$$

$[G],[T]$ and $[\Sigma]$ are assembled matrix relating respectively to the displacement kernel , stress kernel and the hyper singular kernel. $[I]$ is the identity matrix.

Validation

Plane wave diffraction by inclusion:

The numerical formulation is validated using analytical solution of plane wave diffraction by cylindrical homogenous inclusion [4].

We present a superposition in polar representation obtained by analytical solution and coupled numerical formulation for radial displacement modulus.

The internal domain is a circular homogenous or square homogenous inclusion. In the Figure 2a more important back scattering is shown in the case of plane interface. In Figure 2b circular inclusion is considered, so we notice that diffracted lobes are generated when inclusion stiffness is greater.

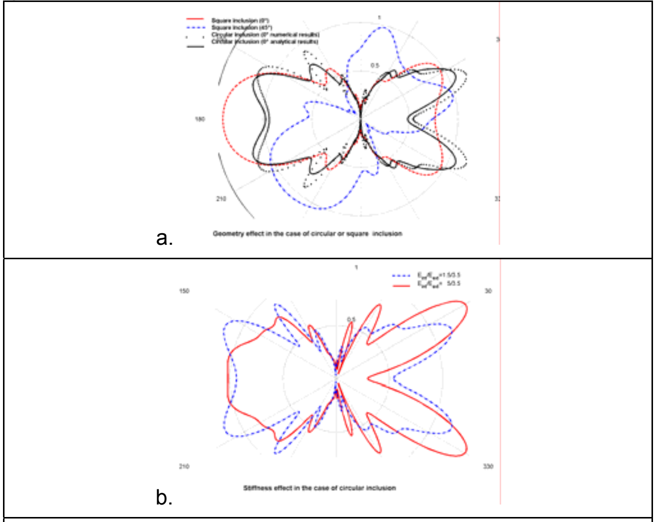

Figure 2. Stiffness and geometry effect for plane wave diffraction.

In Figure 3a we consider orthotropic inclusion (*x, y* principal directions), we notice that the evolution of mean energy flux in *y* direction increases when inclusion stiffness E_y increases. We show also in Figure 3b that for many incidences angle, the mean energy flux is diverted in different directions for fixed elastic properties of the circular inclusion.

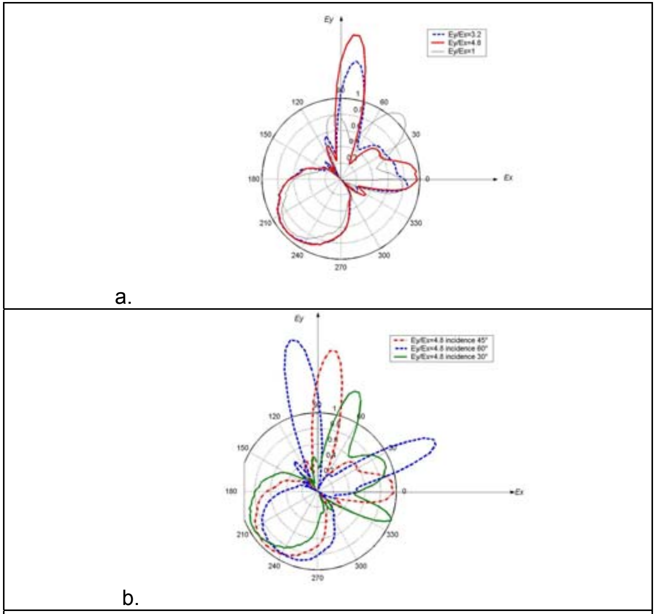

Figure 3. Anisotropic and incidence effect for plane wave diffraction.

Stoneley modal wave problem in interface:

Consider two elastic half spaces in perfect contact. The usual equation of motion is resolved to find modal solution for wave propagating in the *x* and *y* directions that vanishes when *y* increases [5]. For the real roots of the dispersion equation, we find (numerically) the wave number solution for a given frequency *f*. The modal solution is then injected as boundary condition on coupled domains. We compare in Figure 4, numeric and analytic displacements given in the two domains. This kind of mode is shown in the case of plane wave diffraction, independently of the chosen frequency, and of the inclusion geometry, we noticed the generation of this type of modal wave on the interface.

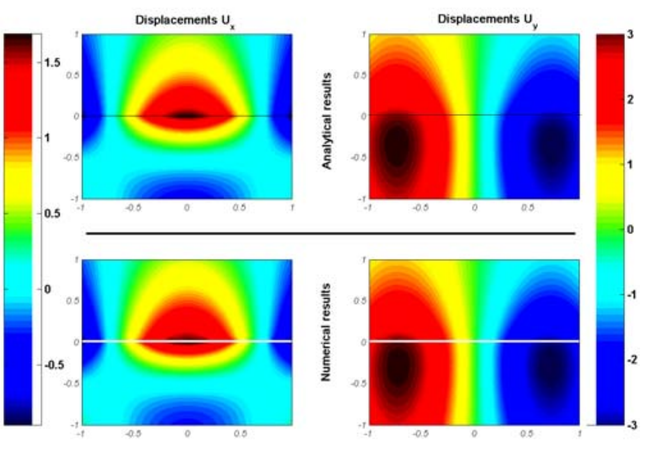

Figure 4. Stoneley modal case validation for the interface Alu/Tung.

Conclusion

We present in this work a numerical coupled formulation by FEM and BEM applied to interface problem and wave transmission. This formulation is validated for plane wave diffraction using analytical solution. We discussed effects of stiffness, geometry and incidence angle on wave diffraction. We show interface wave generated in this case, so we have given for the case of Stoneley mode a numerical validation by the coupled method. The coupling method gives very accurate results for low meshing criterion.

References

[1]. M. Costabel, E.P. Stephan , Coupling of finite elements and boundary elements for transmission problems of elastic waves in R3, in IUTAM Symposium on Advanced Boundary Element Methods, T.A. Cruse, ed., Springer-Verlag, Berlin, New York, pp. 117-124.

[2]. Polizotto C, Zito M, Variational formulation for coupled BE/FE methods in elastostatics, in ZAMM Z angew. Math. Mech. **74** (1994), 533 – 543.

[3]. Han Hou-de, A new variational formulation for the coupling of finite and boundary element methods, in Jour. Of Comp. Mathe., **8**, pp 223-232 (1990).

[4]. A. Cemal Eringen, Erdogan S. Suhubi. Linear theory. ACADEMIC PRESS New York. San Fransisco London 1975.

[5]. Rose, J. L. Ultrasonics Waves in Solid Media, Cambridge University Press, 1999.