Two-side distribution function and WKB solutions for ultrasound at stratified gas

Maxim Solovchuk[1], Sergey Leble[2]
[1] I.Kant State University of Russia, 236041 Kaliningrad, Russia, Email: solovchuk@yandex.ru
[2] Gdansk University of Technology, 80-952 Gdansk, Poland, Email: leble@mif.pg.gda.pl

Introduction

Recently the problems of Kn regime wave propagation was revisited in connection with general fluid mechanics and nonsingular perturbation method development [1, 2, 3]. A generalized Boltzman theories [4] also contributed in a progress with respect to this important problem.

In [5] the propagation of one-dimension disturbance was studied on the base of the method of a piecewise continuous distribution function launched in a pioneering paper of Lees [6] and applied for a gas in gravity field in [1, 7]. We derived hydrodynamic-type equations for a gas perturbations in gravity field so that the Knudsen number depends on the (vertical) coordinate. The generalization to three dimensions is given at [8].

The derivation of the hydrodynamic-type equations is based on kinetic equation with the model integral of collisions in BGK (Bhatnagar - Gross - Krook) form which collision term is modelled as $\nu\,(f_l - f)$, via local-equilibrium distribution function f_l and the non-equilibrium one is expressed as f^+ at $v_z \geq 0$, and as f^- at $v_z \leq 0$

$$f^{\pm} = \frac{n^{\pm}}{\pi^{3/2} v_T^{\pm 3}} \exp\left(-\frac{(\vec{V} - \vec{U}^{\pm})^2}{v_T^{\pm 2}}\right),$$

the $v_T = \sqrt{2kT/m}$ denotes the average thermal velocity of particles of gas, $\nu = \nu(z)$ is the effective frequency of collisions between particles of gas at height z. It is supposed, that density of gas n, its average speed $\vec{U} = (u_x, u_y, u_z)$ and temperature T are functions of time and coordinates enter the local-equilibrium $f_{ l}$.

The increase of the number of parameters of distribution function results in that the distribution function differs from a local-equilibrium one and describes deviations from hydrodynamical regime. In the range of small Knudsen numbers $l << L$ we automatically have $n^+ = n^-, U^+ = U^-, T^+ = T^-$ and distribution function reproduces the hydrodynamics of Euler and at the small difference of the functional 'up' and 'down' parameters - the Navier-Stokes equations. In the range of big Knudsen numbers the theory gives solutions of collisionless problems [7].

We restrict ourselves by the case of one-dimensional disturbances $\vec{U} = (0, 0, U)$. We used such set of linearly independent functions:

$$\begin{array}{lll} \varphi_1 = m & \varphi_2 = mV_z & \varphi_3 = \frac{1}{2} m\xi^2 \\ \varphi_4 = m\xi_z^2 & \varphi_5 = \frac{1}{2} m\xi_z\xi^2 & \varphi_6 = \frac{1}{2} m\xi_z^3 \end{array} \qquad (1)$$

where $\vec{\xi} = \vec{V} - \vec{U}$ is the peculiar velocity. Let's define a scalar product in velocity space:

$$< \varphi_n, f > \equiv < \varphi_n > \equiv \int d\vec{v}\, \varphi_n f \ . \qquad (2)$$

$$\begin{array}{lll} < \varphi_1 > = \rho \ , & < \varphi_2 > = \rho U \ , & < \varphi_3 > = \frac{3}{2}\frac{\rho}{m} kT \ , \\ < \varphi_4 > = P_{zz} \ , & < \varphi_5 > = q_z, & < \varphi_6 > = \bar{q}_z. \end{array} \qquad (3)$$

Here ρ is mass density, P_{zz} is the diagonal component of the pressure tensor, q_z is a vertical component of a heat flux vector, $\bar{q}_z$ is a parameter having dimension of the heat flux.

If we now multiply the kinetic equation with the model integral of collisions in BGK form by φ_i and integrate over velocity space, we obtain the fluid dynamic equations

$$\begin{array}{l}
\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial z}(\rho U) = 0 \ , \\
\frac{\partial}{\partial t}U + U\frac{\partial}{\partial z}U + \frac{1}{\rho}\frac{\partial}{\partial z}P_{zz} + g = 0 \ , \\
\frac{3}{2}\frac{k}{m}\frac{\partial}{\partial t}(\rho T) + \frac{3}{2}\frac{k}{m}U\frac{\partial}{\partial z}(\rho T) + (\frac{3}{2}\frac{k}{m}\rho T + \\
+P_{zz})\frac{\partial}{\partial z}U + \frac{\partial}{\partial z}q_z = 0 \ , \\
\frac{\partial}{\partial t}P_{zz} + U\frac{\partial}{\partial z}P_{zz} + 3P_{zz}\frac{\partial}{\partial z}U + 2\frac{\partial}{\partial z}\bar{q}_z = \\
= -\nu(z)(P_{zz} - \frac{\rho}{m}kT) \ , \\
\frac{\partial}{\partial t}q_z + U\frac{\partial}{\partial z}q_z + 2(q_z + \bar{q}_z)\frac{\partial}{\partial z}U - \\
-(\frac{3}{2}\frac{k}{m}T + \frac{1}{\rho}P_{zz})\frac{\partial}{\partial z}P_{zz} + \frac{\partial}{\partial z}J_1 = -\nu(z)q_z \ , \\
\frac{\partial}{\partial t}\bar{q}_z + U\frac{\partial}{\partial z}\bar{q}_z + 4\bar{q}_z\frac{\partial}{\partial z}U - \frac{3}{2\rho}P_{zz}\frac{\partial}{\partial z}P_{zz} + \\
\frac{\partial}{\partial z}J_2 = -\nu(z)\bar{q}_z \ ,
\end{array} \qquad (4)$$

where

$$J_1 = \frac{m}{2} < \xi_z^2 \xi^2 > \ , \quad J_2 = \frac{m}{2} < \xi_z^4 > \ . \qquad (5)$$

The system (4) of the equations according to the derivation scheme is valid at all frequencies of collisions and within the limits of the high frequencies should transform to the hydrodynamic equations.

We shall base here on an expansion in small Mach numbers $M = max|\frac{U}{v_T}|$, up to the first order. Let's evaluate the integrals (3) directly, plugging the two-side distribution function. Solving this system, we obtain the parameters of the two-fold distribution function as functions of thermodynamic ones.

The values of integrals (5) as functions of thermodynamic parameters of the system (4) are:

$$\begin{aligned} J_1 &= -\frac{5}{2}\rho\left(\frac{kT_0}{m}\right)^2 + \frac{11}{4}\frac{kT_0 P_{zz}}{m} + \frac{9}{4}\left(\frac{k}{m}\right)^2\rho T_0 T, \\ J_2 &= -\frac{3}{2}\rho\left(\frac{kT_0}{m}\right)^2 + \frac{9}{4}\frac{kT_0 P_{zz}}{m} + \frac{3}{4}\left(\frac{k}{m}\right)^2\rho T_0 T. \end{aligned} \quad (6)$$

So we have closed the system (4), hence a modification of the procedure for deriving fluid mechanics equations from the kinetic theory is proposed, it generalizes the Navier-Stokes at arbitrary density (Knudsen numbers). Our method gives a reasonable agreement with the experimental data in the case of homogeneous gas [5]. In the paper [5] the expressions for $(J_{1,2})$ are obtained with account some nonlinear terms, that finally lead to more exact results.

Construction of solutions of the fluid dynamics system by WKB method.

We linearize the fluid equations (3). In this section we apply the method WKB to the linearized system. We shall assume, that on the bottom boundary at $z = 0$ a wave with characteristic frequency ω_0 is generated. Next we choose the frequency ω_0 to be large enough, to put characteristic parameter $\xi = \frac{3\omega_0 H}{v_T} \gg 1$. We shall search for the solution in the form:

$$M_n = \psi_n \exp(i\omega_0 t) + c.c.\ , \quad (7)$$

where, for example, ψ_1, corresponding to the moment M_1, is given by the expansion:

$$\psi_1 = \sum_{k=1}^{6}\sum_{m=1}^{\infty} \frac{1}{(i\xi)^m} A_m^{(k)} \exp(i\xi\varphi_k(z))\ , \quad (8)$$

here $\varphi_k(z)$ - the phase functions corresponding to different roots of dispersion relation. For other moments $M_n\ ,\ \ n = 2\dots6$ corresponding functions ψ_n are given by similar to (8) expansion. The appropriate coefficients of the series we shall designate by corresponding $B_m^{(k)}\, C_m^{(k)}\, D_m^{(k)}\, E_m^{(k)}\, F_m^{(k)}$. Substituting the series (8) at the linearized system one arrives at algebraic equations for the coefficients of (8) in each order. The condition of solutions existence results in the mentioned dispersion relation:

$$\begin{aligned} &\tfrac{54}{125}\eta^3 + \left(-iu^3 + \tfrac{37}{5}iu - \tfrac{24}{5}u^2 + \tfrac{18}{5}\right)\eta + \\ &\left(-\tfrac{12}{5}iu - \tfrac{63}{25} + \tfrac{3}{5}u^2\right)\eta^2 - 1 - 3iu + 3u^2 + iu^3 = 0 \end{aligned} \quad (9)$$

Here for convenience the following designations are introduced:

$$\left(\frac{\partial\varphi_k}{\partial\bar{z}}\right)^2 = \frac{2}{15}\eta_k\ , \qquad u = \frac{\nu_0}{\omega_0}\exp(-\bar{z})\ , \qquad \bar{z} = \frac{z}{H}.$$

The solution of the equation (9) at any u is evaluated numerically. As an illustration let us consider a problem of generation and propagation of a gas disturbance, by a plane oscillating with a given frequency ω_0. We restrict ourselves by the case of homogeneous gas, because it is the only case of existing experimental realization. We evaluate numerically the propagation velocity and attenuation factor of a linear sound.

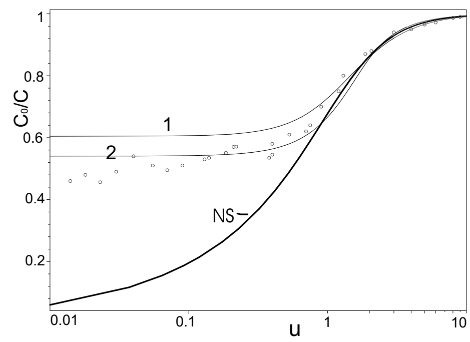

Abbildung 1: The inverse non-dimensional phase velocity as a function of the inverse Knudsen number. The results of this paper-1 are compared to Navier-Stokes, previous our work [5]-2 and the experimental data of Meyer-Sessler [9]-circle.

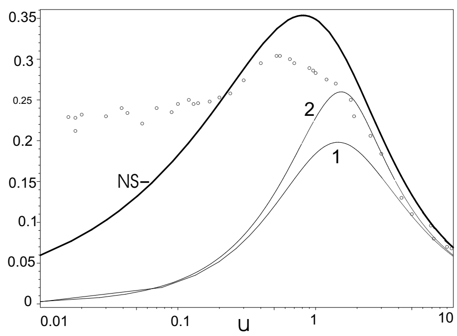

Abbildung 2: The attenuation factor of the linear disturbance as a function of the inverse Knudsen number.

Literatur

[1] D.A. Vereshchagin, S.B. Leble. 1996, *Nonlinear Acoustics in Perspective,ed. R.Wei* , 142-146.

[2] X. Chen, H. Rao, and E. A. Spiegel, 2001, Phys. Rev. E **64**, 046309.

[3] E.A. Spiegel and J.-L. Thiffeault, Physics of Fluids 15 11, (2003), 3558-3567.

[4] B.V. Alexeev, Generalized Boltzmann Physical Kinetics, Elsevier,2004.

[5] D.A. Vereshchagin, S.B. Leble, M.A. Solovchuk. Phys. Let. A,348 (2006) ,326-334.

[6] L. Lees, J.Soc.Industr. and Appl.Math.,13.№ 1, (1965). 278-311.

[7] D.A. Vereshchagin, S.B. Leble, (2005) physics/0503233.

[8] Solovchuk M.A., Leble S.B. Proc. Intern. Conf. 'Forum Acusticum 2005'(Budapest, 2005) L235-L240.

[9] E. Meyer, G. Sessler, Z.Physik. 149. (1957). P.15-39.