Numerical solution of the one–dimensional wave equation with the stochastic parameters using the generalized polynomial chaos expansion

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Introduction

Herein, we develop a method based on the application of a vibrating string; however, the method is simply applicable for other cases which have been mentioned above. A very simple but reliably performing probabilistic method to solve for stochastic PDEs consists in Monte Carlo simulation. It is quite robust and underlies hardly any limitations with respect to the nature of the uncertainties and their distribution. The serious drawback of Monte Carlo simulation methods consists in their computational inefficiency, cf. [1]. An alternative approach is referred to as stochastic spectral method. There, a spectral representation of uncertainties in terms of a multi–dimensional polynomial chaos (PC) expansion is used. The most significant property of PC is the possibility to analytically manage any probability distribution in closed form. The solution is considered as an element of the Hilbert space of random functions. It is approximated by its projection into a finite series of orthogonal polynomials,e.g. Hermite polynomials, [2].

Polynomial Chaos (PC) expansion

The PC was first introduced as the homogeneous chaos expansion by Wiener [3]. The basic idea is to project the variables of the problem onto a stochastic space spanned by a set of complete orthogonal polynomials Ψ which are functions of a random variable $\boldsymbol{\xi}$. The random variable $\boldsymbol{\xi}$ is a multidimensional random variable as $\boldsymbol{\xi} = \{\xi_1, \xi_2, ..., \xi_n\}$. Using this approach, the random variable $\boldsymbol{x}$ is expanded as [4]

$$\begin{aligned} \boldsymbol{x} &= x_0\Psi_0 + \sum_{i_1=1}^{\infty} x_{i_1}\Psi_1(\xi_{i_1}) + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1} x_{i_1 i_2}\Psi_2(\xi_{i_1},\xi_{i_2}) \\ &+ \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2} x_{i_1 i_2 i_3}\Psi_3(\xi_{i_1},\xi_{i_2},\xi_{i_3}) + \dots \end{aligned} \tag{1}$$

or, shortly,

$$\boldsymbol{x} = \sum_{i=0}^{\infty} x_i \Psi_i(\boldsymbol{\xi}) \ , \tag{2}$$

where x_i are deterministic unknown coefficients. They will be denoted as random modes of the system. The random base functions Ψ_i are a set of complete multidimensional polynomials in terms of the multidimensional random variable $\boldsymbol{\xi}$ with the orthogonality relation of

$$\boldsymbol{E}[\Psi_i, \Psi_j] = \boldsymbol{E}[\Psi_i^2]\delta_{ij} \ , \tag{3}$$

where δ_{ij} represents the Kronecker delta and $\boldsymbol{E}$ denotes the expectation value with respect to the probability space. The unknown coefficients, x_i, can be determined by

$$x_i = \frac{1}{\langle \Psi_i^2 \rangle} \int_{\Omega} \boldsymbol{x}\Psi_k(\boldsymbol{\xi})\rho(\boldsymbol{\xi})d\boldsymbol{\xi} \ , \quad k = 1, 2, \dots \ , \tag{4}$$

where ρ is a weight function corresponding to the random variable $\boldsymbol{\xi}$. Ω stands for a sample space corresponding to the probability space.

PC expansion of the stochastic vibrating string equation

We consider the problem of a vibrating string as an application of 1–D wave equation. A stretched string of length l has a mass per unit length of m and a tension of T, cf. Fig. 1. In the case of uncertain parameters m and T, the equation of vibrating string can be written as

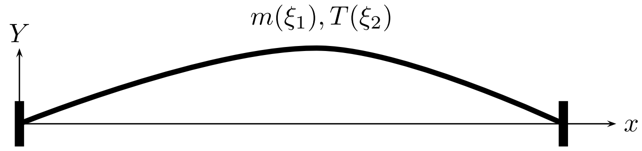

Figure 1: String with the stochastic parameters

$$m(\xi_1)\frac{\partial^2 Y(x,t;\boldsymbol{\xi})}{\partial t^2} = T(\xi_2)\frac{\partial^2 Y(x,t;\boldsymbol{\xi})}{\partial x^2}. \tag{5}$$

Subjected to the corresponding boundary and initial conditions. The vector $\boldsymbol{\xi} = \{\xi_1, \xi_2\}$ contains the random variables. We assume that m and T are uncertain parameters with the truncated PC expansions as

$$m(\xi_1) = \sum_{k=0}^{N_m} \mathfrak{m}_k \Psi_k(\xi_1) \qquad T(\xi_2) = \sum_{j=0}^{N_T} \tau_j \Psi_j(\xi_2) \tag{6}$$

Frequency of the string vibration with the uncertain parameters

For a string which is fixed at both ends, the resonance frequency of the vibration can be derived as

$$f_n^2(\boldsymbol{\xi}) = \frac{n}{4l^2}\frac{T(\xi_2)}{m(\xi_1)}, \qquad n = 1, 2, \dots, \infty \tag{7}$$

We represent the PC expansion of fundamental frequency as

$$f_n^2(\boldsymbol{\xi}) = \sum_{i=0}^{N_f} \mathfrak{f}_{n_i} \Psi_i(\xi_1, \xi_2) \tag{8}$$

Where $\Psi_i(\xi_1, \xi_2)$ is PC orthogonal basis which can be derived from the tensor product of $\Psi_k(\xi_1)$ and $\Psi_j(\xi_2)$. The coefficients of PC expansion for fundamental frequency, $\mathfrak{f}_{n_i}$,then can be derived by substitute the Eqs. (6) and (8) in Eq. (7) by using Galerkin projection as

$$\sum_{i=0}^{N_f} \sum_{k=0}^{N_m} \mathfrak{f}_{0_i} \mathfrak{m}_k e_{ikm} = \frac{1}{4l^2} \sum_{j=0}^{N_T} \tau_j e_{jm}, \qquad m = 0, 1, \ldots, N_f \tag{9}$$

In which $e_{ikm} = \langle \Psi_i(\boldsymbol{\xi})\ \Psi_k(\xi_1)\ \Psi_m(\boldsymbol{\xi}) \rangle$ and $e_{jm} = \langle \Psi_j(\xi_2)\ \Psi_m(\boldsymbol{\xi}) \rangle$. The Eq. (9) is the deterministic linear system of equations of order $(N_f + 1) \times (N_f + 1)$ and can be solved for determination of coefficients $\mathfrak{f}_{0_i}$.

String vibration modes

We assume that $Y(x, t; \boldsymbol{\xi})$ is a second–order process with a finite variance. Then, the truncated general polynomial chaos representation of $Y(x, t; \boldsymbol{\xi})$ is given by

$$Y(x, t; \boldsymbol{\xi}) = \sum_{i=0}^{N_y} y_i(x, t) \Psi_i(\boldsymbol{\xi}) \ , \tag{10}$$

where $y_i(x, t)$ are deterministic unknown functions. Substitute the truncated PC expansions for m, T and Y in Eq. (5), leads to

$$\sum_{j=0}^{N_T} \tau_j \Psi_j(\boldsymbol{\xi}) \sum_{i=0}^{N_y} \frac{\partial^2 y_i(x, t)}{\partial x^2} \Psi_i(\boldsymbol{\xi}) = \sum_{k=0}^{N_m} \mathfrak{m}_k \Psi_k(\boldsymbol{\xi}) \sum_{i=0}^{N_y} \frac{\partial^2 y_i(x, t)}{\partial t^2} \Psi_i(\boldsymbol{\xi}) \tag{11}$$

Now, we multiply both sides of the above equation by the test function $\Psi_p(\boldsymbol{\xi})$ to achieve a Galerkin projection. It can be rewritten as a general form of

$$\boldsymbol{T} \frac{\partial^2 \boldsymbol{y}(x, t)}{\partial x^2} = \boldsymbol{M} \frac{\partial^2 \boldsymbol{y}(x, t)}{\partial t^2} \tag{12}$$

In which $\boldsymbol{y} = \{y_0, y_1, \ldots, y_j\}$, $\boldsymbol{M}$ and $\boldsymbol{T}$ are mass and tension matrices respectively, defined as

$$\boldsymbol{T} = \sum_{i=0}^{N_y} \sum_{j}^{N_T} \tau_j e_{jip} \quad \boldsymbol{M} = \sum_{i=0}^{N_y} \sum_{k}^{N_m} \mathfrak{m}_k e_{kip} \quad p = 0, 1, \ldots, N_y \tag{13}$$

Equation (13) define a system of second order linear partial differential equations (pde) with the projected initial conditions and boundary condition. It can be solved by using appropriate numerical algorithms(FEM, FDE).

Numerical results

Let now attempt to solve for a simple vibrating string problem. Suppose that the initial conditions are independent of the random variable vector, $\boldsymbol{\xi}$, and $Y(x, 0) = 0$, $\dot{Y}(x, 0) = sin(\pi x)$ and $0 \leq x \leq 1$. The string parameters, i.e. T and m, are considered as uncertain parameters with Lognormal and uniform distributions respectively with density functions of

$$\rho_1(T) = \frac{1}{\sigma_T \sqrt{2\pi} T} e^{\frac{-(\ln(T) - \mu_T)^2}{2\sigma_T^2}}, \quad \rho_2(m) = \frac{1}{m_{min} - m_{max}} \tag{14}$$

With $(\mu_T, \sigma_T) = (2, 0.25)$ and $(m_{min}, m_{max}) = (4, 7)$ kg/m. We use the 4^{th} order Hermite-PC and 1^{th} order Legendre-PC for T and m respectively. Table 1 shows the calculated coefficients. As this table shows, the coefficients of PC expansion of random parameter T converge faster than m. So, the PC expansion of order 3 has been considered for T and higher orders are ignored. The PC representation of the PDFs of the first 10 string frequencies are shown in the Fig 2–(a) while the coefficients of the PC representation of the string mode shape at $t = 1$ s have been shown in the FigFig 2–(b).

	m_0, T_0	m_1, T_1	m_2, T_2	m_3, T_3	m_4, T_4
T	2.8046	0.7011	0.0876	0.0074	0.0005
m	5.5	1.50	0.0	0.0	0.0

Table 1: The coefficients of the PC expansions for the string uncertain parameters ,T and m

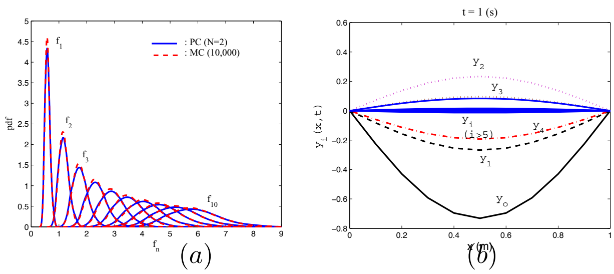

Figure 2: (a)–PDFs of the first 10 string frequencies and (b)– PC coeffs. $y_i(x, t)$ in the Eq. (10)

Conclusions and remarks

In this paper we have extended the application of polynomial chaos expansion to solve the partial differential equations with stochastic parameters. It has been shown that the generalized polynomial chaos can be orders of magnitude more efficient than Monte Carlo simulations. However, there are still some unresolved problems for the generalized polynomial chaos and more work is required to resolve these issues.

References

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