Bending waves and spatially varying structural properties

Björn A.T. Petersson

Institut für Strömungsmechanik und Technische Akustik, TU-Berlin, E-Mail b.a.t.petersson@tu-berlin.de

Introduction

With an increased emphasis on light-weight and cost efficient designs, spatially varying material properties or geometry have become important aspects for many engineering applications. This trend has particular implications for the structural acoustic performance which often is directly governed by the dynamic characteristics of a wave carrying structure and where, for example, mass reductions implies enhanced vibrations.

A physically correct description of structural acoustic behaviour of built-up systems can often be obtained using wave theoretical models involving elementary systems, which are combined in such a way that the governing physical phenomena of the real system are represented. Such models offer the analyst and designer versatile tools as long as structural elements with constant parameters can be assumed. For optimization purposes, however, spatially varying structural parameters are attractive but the structural acoustic significance and influence of such variations are insufficiently clarified.

The flexural vibration of beams and plates with varying cross-section has received substantial attention from the work of Kirchhoff and onwards (see e.g. [1]) but the structural acoustic aspects are not considered correspondingly. Based on a wave theoretical consideration for structures having spatially varying parameters, some of the implications are herein exemplified for the practically important case of a linearly tapered beam [2].

Theory

According to Euler-Bernoulli beam theory, the equation of motion of a beam in bending with harmonic time dependence reads,

$$\frac{d^2}{dx^2} B \frac{d^2 v}{dx^2} - m' \omega^2 v = 0 \,, \tag{1}$$

If B and m' vary along the beam, exact solutions exist provided B and m' can be expressed by suitable functions of x. Here focus is put on solutions in terms of Bessel functions as they allow for a large assortment of shape and material property variations.

It can be shown [3] that equation (1) has exact solutions in terms of Bessel functions if,

$$m' = \tau_1 x^n, \quad B = \tau_2 x^{n+2} \,, \tag{2}$$

where τ_1 and τ_2 are arbitrary constants. For integer values of n the solution reads,

$$\begin{aligned} v(x) = x^{-n/2} \Big(& c_1 \, \mathrm{J}_n(2\kappa\sqrt{x}) + c_2 \, \mathrm{Y}_n(2\kappa\sqrt{x}) \\ & + c_3 \, \mathrm{I}_n(2\kappa\sqrt{x}) + c_4 \, \mathrm{K}_n(2\kappa\sqrt{x}) \Big) \end{aligned} \,, \tag{3}$$

with $\kappa^4 = \tau_1 \omega^2 / \tau_2$. The requirements expressed by equations (2) may be met in many different ways; for instance by keeping the material properties constant and varying the cross section along the beam. For rectangular cross sections this results in beams which must in any case have a linear depth taper but where the width variation may take on any power of x. Other possibilities are to fix the beam shape and vary the material properties along the beam or else to vary both material properties and cross sectional dimensions. In the case of a beam with a rectangular cross section, a linear depth variation but a non-homogeneous beam material it may easily be verified that the density, ρ and Young's modulus, E, must vary with the same power of x. Bessel function solutions similar to those of equation (3) can also be found for quadratic depth and any power of width variation and for cubic depth and any power of width variation [4]. In the following, the focus is on beams with constant width and a linear depth taper as they are prominent from a practical point of view. They are furthermore expected to exhibit all the basic features related to bending vibrations of variable section beams.

Consider a finite beam with a rectangular cross section of constant width b and a depth which varies linearly between h_0 at x=0 and h_1 at x=L. It is further assumed that the beam is made of a homogeneous, linear elastic material. The depth X of the beam is a function of the longitudinal co-ordinate x and may be expressed as $X(x) = h_0 + \alpha x$ with $\alpha = (h_1 - h_0)/L$. The mass per unit length and the bending stiffness are consequently given by $m' = \rho b X$ and $B = E b X^3/12$ respectively. The four unknown (frequency dependent) coefficients c_1, c_2, c_3 and c_4 can be determined from the four boundary conditions at the beam ends. Closed form expressions can therefore be derived for every single element of the finite tapered beam impedance matrix [2]. This impedance matrix can be formally written as

$$\begin{Bmatrix} F_0 \\ M_0 \\ F_1 \\ M_1 \end{Bmatrix} = \begin{bmatrix} Z_{00}^{Fv} & Z_{00}^{Fw} & Z_{01}^{Fv} & Z_{01}^{Fw} \\ Z_{00}^{Mv} & Z_{00}^{Mw} & Z_{01}^{Mv} & Z_{01}^{Mw} \\ Z_{10}^{Fv} & Z_{10}^{Fw} & Z_{11}^{Fv} & Z_{11}^{Fw} \\ Z_{10}^{Mv} & Z_{10}^{Mw} & Z_{11}^{Mv} & Z_{11}^{Mw} \end{bmatrix} \begin{Bmatrix} v_0 \\ w_0 \\ v_1 \\ w_1 \end{Bmatrix}. \tag{4}$$

Attached to the tapered tip is a semi-infinite beam with uniform cross section. The cross sections at the beam-tip interface are identical. This configuration represents the bending wave equivalent of the acoustic horn, because of the gradual transition between the tip impedance and that of the semi-infinite beam. The impedance matrix describing the bending wave horn is obtained by adding

$$\mathbf{Z}_\infty = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{B_\infty k_\infty^3}{\omega}(1+i) & \dfrac{B_\infty k_\infty^2}{\omega} \\ 0 & 0 & \dfrac{B_\infty k_\infty^2}{\omega} & \dfrac{B_\infty k_\infty}{\omega}(1-i) \end{bmatrix} \qquad (5)$$

to the impedance matrix in equation (4), where k_∞ and B_∞ are the bending wave number and the bending stiffness respectively of the prismatic semi-infinite beam. Note that the characteristic beam impedances in the matrix of (5) consist of the classical Euler Bernoulli expressions which neither account for shear deformation nor rotational inertia implying that the expressions only applies to slender beams. The driving point mobility of the bending wave horn (at the free end) is obtained from a matrix inversion.

For the semi-infinite wedge the following general solution may be assumed for the velocity field,

$$v(X) = \frac{1}{\sqrt{X}}\left(A\,\mathrm{H}_1^{(2)}\left(2\kappa\sqrt{X}\right) + B\,\mathrm{K}_1\left(2\kappa\sqrt{X}\right)\right). \qquad (6)$$

A and B are the unknown (frequency dependent) coefficients to be determined from the boundary conditions at the free end. In Figure 1 is presented the normalized point mobility as function of Helmholtz number $\kappa\sqrt{h_0}$. For small Helmholtz numbers, the mobility is dominated by the imaginary part and it is observed that this is stiffness controlled. This can be explained by the fact that for sufficiently small Helmholtz numbers the depth at the free end becomes very small in comparison with the beam depth at for instance a quarter wavelength from the end, establishing a cantilever type of behaviour of the tip. The point mobility magnitude exhibits a broad maximum centred at $\kappa\sqrt{h_0} \approx 1{,}184$ (the estimate is obtained from close inspection of the imaginary part) while the imaginary part changes sign. At high Helmholtz numbers the mobility of the semi-infinite wedge asymptotically approaches the mass governed mobility of a semi-infinite prismatic beam with depth h_0 i.e., the depth at the free end.

In order to gain more insight into the rather involved expression for the characteristic mobility it is useful to study the asymptotic behaviour for small Helmholtz numbers. Upon expanding the Bessel functions for small arguments, the asymptotic transfer mobility can be found to be given by

$$Y_{vF}(X) \to \sqrt{\frac{12}{E\rho}}\frac{\kappa^2}{b\alpha}\left[\frac{\pi}{2} - 2i\left(\ln\left(\kappa\sqrt{X}\right) + \gamma_E - \frac{1}{2} + \frac{h_0}{4X}\right)\right] \qquad (7)$$

From (7) is found that the real part of the mobility at the tip, and therefore the power fed into the beam by force excitation, is independent of h_0 at low Helmholtz numbers whereas the imaginary part has a logarithmic dependence. Reduction of the tip height, for a constant tapering angle, may therefore result in arbitrarily high driving point mobility while the power remains finite. This may again be explained by the fact that at low Helmholtz numbers the beam depth, "seen" by the propagating wave is much larger than the tip depth; the latter consequently has no significance for the energy but governs the deformation.

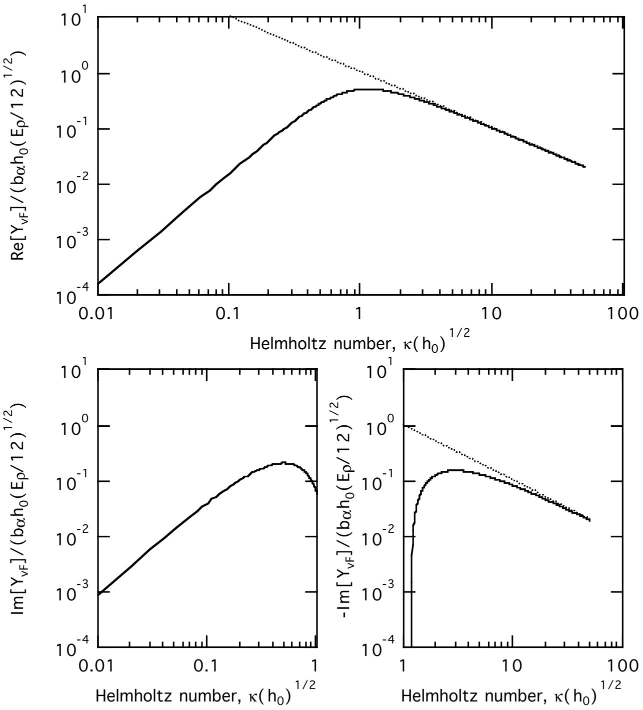

Figure 1: Normalised real and imaginary parts of force mobility at the tip of a semi-infinite wedge. Normalised real and imaginary parts of force mobility of a uniform, semi-infinite prismatic beam of height equal to that of the tip of the semi-infinite wedge (······).

Concluding remarks

From the analysis of the influence of tapering on the dynamic characteristics of the flexural wave counterpart to the acoustic horn, it is found that the main distinction to the uniform case is the comparatively broad-banded transition from flexural vibrations governed by the properties of the deep part of the system to flexural vibrations governed by those of the slender part. The transition itself, furthermore, involves a transition from mainly translational to rotational motion of the tapered part.

Literatur

[1] D.L. Karabalis and D.E. Beskos, 1983. Computers and structures, 16, 731-748..

[2] B.A.T. Petersson and E.J.M. Nijman, 1998. Journal of Sound and Vibration, 211, 95-121.

[3] E.T. Cranch and A.A. Adler, 1956. Journal of Applied Mechanics, vol.23, 103-108.

[4] E.J.M. Nijman and B.A.T. Petersson, 2000. International Journal of Acoustics and Vibration, 5 (2).