Frames and Acoustic BEM

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Introduction

The problem of sound propagation in a homogeneous, isotropic, friction-free medium is governed by the *Helmholtz equation*

$$-\left(\Delta+\kappa^2\right)u(x)=\mathcal{L}u(x)=f(x) \quad \text{with} \quad x\in\Omega\ , \quad (1)$$

where $\kappa\in\mathbb{C}$, $\mathfrak{Im}(\kappa)\geq 0$ is the complex wave number, $u:\mathbb{R}^3\to\mathbb{C}$ is the acoustic pressure field, $f:\mathbb{R}^3\to\mathbb{C}$ is the excitation force which has compact support and Ω is a bounded, multiply or simply connected domain with a Lipschitz boundary Γ.

Multiplying equation (1) with its fundamental solution $u^*_\kappa:\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{C}$,

$$u^*_\kappa(x,y)=\frac{e^{\mathrm{i}\kappa|x-y|}}{4\pi|x-y|}\ , \quad (2)$$

integrating over Ω, and using Green's identity finally leads to the boundary integral equation (BIE)

$$u(x)=\int_\Gamma u^*_\kappa(x,y)v(y)\ ds_y-\int_\Gamma v^*_\kappa(x,y)u(y)\ ds_y\ , \quad (3)$$

where $v(y)=\frac{\partial u(y)}{\partial n_y}$, $v^*_\kappa(x,y)=\frac{\partial u^*_\kappa(x,y)}{\partial n_y}$ and n_y is the outward normal vector at $y\in\Gamma$.

Creating the Linear System

To create a system of linear equations from (3) a discretisation (also called mesh) of the boundary, consisting of N (for example triangular) elements is introduced.

The simplest way of constructing a basis for the solution space now is to introduce constant shape functions which yields constant basis functions, spanning an N-dimensional solution space of piecewise constant functions.

With this projection of the problem into a finite dimensional space one can now actually implement the method into a computer program. However, it has certain limitations, because the matrices involved are dense and the method thus has quadratic memory requirement.

One way to overcome these limitations is to choose an adaptive solution process to control the size of the problem and use (wavelet) frames instead of bases for the solution space to get sparser matrices of the linear system that has to be solved.

Frames

The concept of a frame is a generalisation of a basis. Its formal definition is as follows: A sequence $\{f_k\}_{k\in I}$ in a Hilbert space $\mathcal{H}$ is called a frame for $\mathcal{H}$ if and only if there exist $0<A,B\in\mathbb{R}$ with

$$A\|f\|^2\leq\sum_{k\in I}|\langle f,f_k\rangle|^2\leq B\|f\|^2\ , \quad (4)$$

where A and B are called the frame bounds. If $A=B$, the frame is called a tight frame. A frame is called overcomplete if it is not a basis.

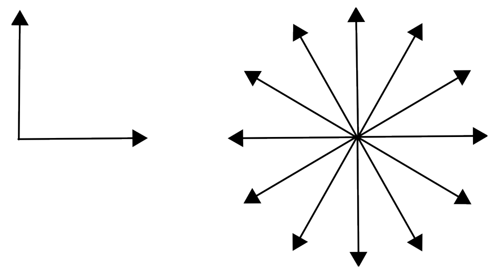

Figure 1: LEFT: Basis for $\mathbb{R}^2$. RIGHT: overcomplete frame for $\mathbb{R}^2$. This frame is even a tight frame, i.e. $A=B$.

Despite the overcompleteness of frames, using them instead of bases can have the advantages of sparse representation and greater error tolerance. A way to construct a frame for the solution space of the BEM is to use frames on the reference domain [2]. A different way is to describe Γ with an overlapping domain decomposition [11].

Adaptivity and Optimality

We plan to solve the linear system with an iterative, adaptive method. The idea for this method stems from [3], where it is shown how a damped Richardson iteration for an infinite matrix-vector system (that was obtained by transforming the elliptic operator equation into the coordinates of a wavelet basis [1] with certain properties derived from the theory of nonlinear approximation [7]) can be solved in an optimal way. In this context, optimal means that the solution is obtained in $\mathcal{O}(N)$, where N is the minimal number of coefficients for which the (finite dimensional) approximation to the solution has at least a given accuracy.

This idea was also shown to work when the linear system is obtained in the coordinates of a wavelet frame, rather than those of a basis [11].

A different iteration method with the advantage of not having to choose the dampening parameter is discussed in [4].

For both iteration methods, two concepts are very important. The first is the computability of a matrix (sometimes called 'compressibility', for a discussion of computability versus compressibility see [9, p. 79]) that describes how well the bi-infinite matrix can be approximated by a finite one. The second is the approximative sorting algorithm called binary binning, explained in [11, p. 1084].

That the matrices occurring for differential operators (such as the Helmholtz operator) are indeed computable in this sense was demonstrated in [8]. The proof of the corresponding property for operators with singular kernels, such as those occuring in the boundary element method, can be found in [9].

In [5] the notion of Gelfand frames is introduced and shown to be a suitable tool in this field.

First numerical results, obtained by methods similar to the one proposed here can be found in [6].

Application in Acoustics

A possible application of our proposed method is the calculation of head-related transfer functions (HRFTs).

The shape of head, torso, and pinna play an important role in the localisation of sounds [10]. Reflections, especially at the pinna, act as a filter which can be described using the HRTFs. These can, in turn, be used to create virtual free field sounds.

High-quality meshes of head, pinna and torso are required to get useful results with numerical simulations of HRTFs as can be seen from the comparison between measured and calculated data in Figure 2. These results were calculated using the fast multipole method. They illustrate the need for methods that can deal with meshes of high quality.

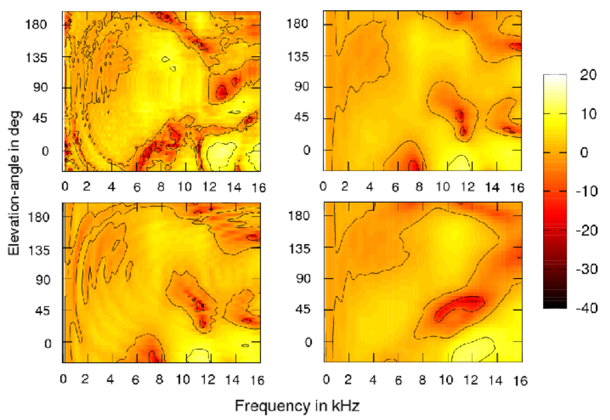

Figure 2: HRTFs for the median plane. TOP LEFT: Measured data. TOP RIGHT: Calculated data using high-quality mesh without torso. BOTTOM LEFT: Calculated data using high-quality mesh with torso. BOTTOM RIGHT: Calculated data using low-quality mesh without torso. [10]

Outlook

Our future work will focus on a rigorous mathematical formulation and the actual implementation of the method discussed here.

Furthermore we plan to investigate the possibility of finding preconditioners and work on the question how the overcompleteness of frames and an according solution scheme help to avoid problems that usually require methods like the Burton-Miller BEM.

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