Inverse calculation of blocked forces in the time domain

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Introduction

For characterisation of structure-borne sound sources often the dynamic forces acting at the interface between an operational vibration source and a passive receiver structure are of interest. At each point where the source and receiver connect there can exist up to three orthogonal forces and three moments about these orthogonal axes. However, the forces acting at the interface usually can not be measured directly, since the presence of the required instruments integrated into the connection may alter the effective forces. Therefore it is often preferable to obtain the forces using inverse methods.

Independent source characterisation

Since operational contact forces are not suitable for prediction of the response when the passive receiver structure differs in its properties, an in situ measurement method which yields independent characteristics of the source activity was developed [1]. Therefore, a two-stage measurement of the vibrating source coupled to a passive receiver structure is required to obtain its blocked forces. In a first measurement, the Frequency Response Function (FRF) matrix $Y_{C,cb}$ containing the generalised transfer mobility of the coupled structure C, excited at the contact interface c and measured at b on the receiver structure is determined. In a second step the vibration source is operated and the responses a_b at the receiver interface b are measured. The source activity can then be determined by the blocked forces f_{bl} where ω denotes the radian frequency:

$$f_{bl}(\omega) = Y_{C,cb}(\omega)^{-1} a_b(\omega) \qquad \text{[N]} \qquad (1)$$

To determine the blocked forces f_{bl} in Eq. (1), the matrix Y_C need to be inverted at each frequency. Therefore, the obtained forces can contain errors due to ill-conditioning of the FRF matrix particularly at frequencies where the condition number is high as well as measurement errors in the operational responses and FRF's can occur. To overcome the ill-conditioned nature of the inverse problem, matrix regularisation methods like Tikhonov regularisation or Singular value rejection can be used. In this paper a novel formulation is presented, with which the solution of the force and the inverse problem can be handled in a time domain representation. An example is shown for a system with a single input excitation with a single output response.

Theory

Considering a single degree of freedom input-output system (SISO), Eq. (1) can be rewritten in matrix formulation, where $\tilde{f}$ and $\tilde{a}$ are vectors containing the double sided Fourier spectra of the force and the response, respectively. These vectors are linked by a diagonal matrix $\tilde{H}$ containing the complex FRF of the system on its main diagonal. In order to transform this equation into time domain, a Fourier component matrix E, whose columns e_n for $m, n = \mathbf{0}$ to $N-\mathbf{1}$ represents the Fourier components of dimension N, can be used:

$$e_n = \frac{\mathbf{1}}{\sqrt{N}} e^{j\mathbf{2}\pi mn/N}, \; E = [e_{\mathbf{0}}, \dots e_{N-\mathbf{1}}] \qquad \text{[-]} \qquad (2)$$

A forward Discrete Fourier Transform of an input or output vector is performed by pre-multiplication of E. An inverse DFT is performed in Eq. (3) by a pre-multiplication of the hermitian transpose E^H. The input-output relation of the spectra is transformed into the time domain by

$$E^H \tilde{a} = \left(E^H \tilde{H} E\right) E^H \tilde{f} \qquad \text{[m/s}^2\text{]} \qquad (3)$$

The input-output relation in time domain is coupled by a cyclic convolution matrix that effectively connects the frequency elements of the FRF. Each column of the obtained matrix H in Eq. (4) contains the Impulse Response Function (IRF) of the system and repeats by a circular down shift of the previous column.

$$f(t) = H(t)^{-\mathbf{1}} a(t) \qquad \text{[N]} \qquad (4)$$

According to Eq. (4) the square cyclic convolution matrix H has to be inverted to obtain the unknown force. For inversion of the convolution matrix the singular value decomposition (SVD) is used and can be written as a decomposition of the form

$$H = U \Sigma V^H = \sum_{i=\mathbf{1}}^{n} u_i \sigma_i v_i^H \qquad \text{[-]} \qquad (5)$$

where u_i and v_i are the left and right singular value vectors of H, respectively. The numbers σ_i are the corresponding non-negative singular values, arranged in descending order. For a SISO system the singular values of the convolution matrix H behave similar to the singular values of the diagonal FRF matrix $\tilde{H}$ used in Eq. (3). Especially if the response or the IRF are contaminated with noise e.g. due to inevitable measurement errors, a least square solution using the Moore-Penrose Pseudo Inverse can be error-prone. Therefore regularisation methods [2] can filter out the worse effects of the small singular values that become large after the inversion. Here a Tikhonov filter factor k_i is used for regularisation:

$$f_\lambda = \sum_{i=1}^{n} k_i \frac{u_i^T a}{\sigma_i} v_i \qquad \text{[N]} \qquad (6)$$

where the filter factors $k_1,...,k_n$ are a function of σ_i and the regularisation parameter λ:

$$k_i = \frac{\sigma_i^2}{\sigma_i^2 + \lambda^2} \simeq \begin{cases} 1, & \sigma_i \gg \lambda \\ \sigma_i^2/\lambda^2, & \sigma_i \ll \lambda \end{cases} \qquad \text{[-]} \qquad (7)$$

The filter factor k_i in Eq. (7) filters out the contributions to f_λ in Eq. (6) corresponding to the small singular values while the SVD components of the large singular values are nearly unaffected. Firstly, for determining the regularisation parameter an idealised method called the minimum force error (MFE) is used. This method uses the fact, that the original force is known to find the best regularisation parameter which minimises the force error f_{err}. For all valid regularisation parameters from $\sigma_{\min}$ to $\sigma_{\max}$ the mean square error of the predicted force f_λ and the original force f_{orig} is determined, and the parameter which yields the minimum, such force error is selected. This shows the best possible solution that can be achieved by regularisation, however it is not intended for practical applications.

$$f_{err}(\lambda) = \frac{1}{n} \sum_{i=1}^{n} \left(f_\lambda(i) - f_{orig}(i) \right)^2 \qquad \text{[-]} \qquad (8)$$

For real applications, unfortunately however the force is not available and therefore an independent method for choosing the best regularisation parameter λ is necessary. Here, the Generalised Cross Validation (GCV) is used to estimate an appropriate value for the regularisation parameter. The GCV function which varies with its regularisation parameter λ is given by:

$$GCV(\lambda) = \frac{\left\| H f_\lambda - a \right\|_2^2}{trace\left(I - HH^{\#} \right)^2} \qquad \text{[-]} \qquad (9)$$

where the numerator is the squared residual norm, I is the identity matrix and $H^{\#}$ is the matrix which produces the regularised solution when multiplied with a. The denominator can also be considered as the effective number of degrees of freedom [2]. The GCV seeks to locate the transition point when the residual norm changes from a slow varying to a rapidly increasing function of λ. The denominator increases monotonically as a function of λ, such that the GCV function has a minimum in the transition interval of the residual norm.

Validation by experimental measurement

In order to validate the method in terms of a measurement an experiment was carried out. A free-free supported beam was used to measure the time history of a single impact hammer excitation and a single response. The convolution matrix in Eq. (4) was determined by measurement from a separate impact hammer hit to avoid circle-calculation. This matrix has been inverted by the Moore-Penrose (MP) Pseudo Inverse and multiplied with the measured time history of the response to achieve a prediction of the hammer force, see Fig.1 a), b). The obtained prediction of the force from the inverted convolution matrix will give exactly the same results as obtained from an inversion frequency by frequency of Eq. (1).

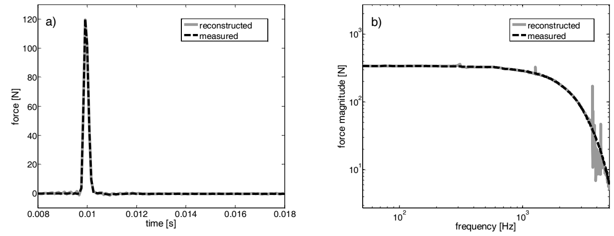

Figure 1: Time history a) and spectrum b) of measured impact hammer force (black) and reconstructed force by use of the MP-Pseudo Inverse (grey).

To show the effect of regularisation in time domain, a white noise signal with a level of 2% of the maximum amplitude of the IRF is added to the convolution matrix. Then the noise contaminated matrix is inverted by the MP-Pseudo Inverse to show the worse effects without regularisation and also the Tikhonov filter factor is applied. Here the regularisation parameter is determined firstly by the MFE and is compared to the parameter determined by the GCV.

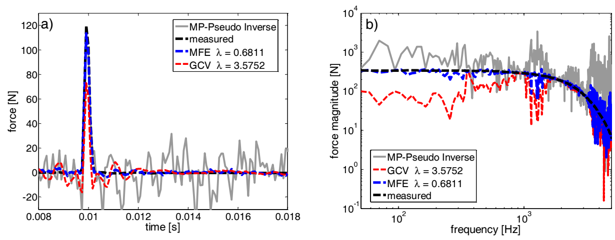

Figure 2: Time history a) and spectra b) of measured force (black), MP-Pseudo Inverse (grey), Tikhonov regularisation MFE: λ = 0.68 (blue) and GCV: λ = 3.57 (red).

Conclusion

It is shown that the inverse problem for obtaining an unknown input force can be solved in the time domain. To improve solution for noise contaminated data, a Tikhonov regularisation filter factor is used. For parameter choice, firstly the MFE is determined. The GCV shows slightly over regularisation to the MFE obtained solution. The input – output relationship using the convolution matrix formulation can also be extended for multiple input - multiple output systems, however this is beyond the scope of this paper.

Literature

[1] A.T. Moorhouse, A.S. Elliott, T.A. Evans. In situ measurement of the blocked force of structure-borne sound sources. Journal of Sound and Vibration 325 (2009) 679-685.

[2] P.C. Hansen. Rank-deficient and discrete ill-posed problems: numerical aspects of linear inversion, SIAM, Philadelphia, 1998.