Asymptotic expansion for viscous acoustic equations close to rigid walls

Anastasia Thöns-Zueva[1], Kersten Schmidt[2],
Institut für Mathematik, Technische Universität Berlin, Deutschland,
[1] *Email: anastasia.thoens@math.tu-berlin.de* [2] *Email: kersten.schmidt@math.tu-berlin.de*

Introduction

In this study we are investigating the acoustic equations as a perturbation of the Navier-Stokes equations around a stagnant uniform fluid, with mean density ρ_0 and without heat flux. For gases the (dynamic) viscosity η is very small and leads to *viscosity boundary layers* close to walls. To resolve the boundary layers with (quasi-)uniform meshes the mesh size has to be at the same order which leads to very large linear systems to be solved. This is especially the case for the very small boundary layers of acoustic waves. We propose effective (*impedance*) boundary conditions for curved boundaries by a multiscale analysis, which separate velocity and pressure into *far field* and correcting *near field*. The boundary conditions are stable and asymptotically exact, which is justified by a complete mathematical analysis in [1] for the components of asymptotic expansion.

Formulation of the problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$. We consider dimensionless time-harmonic acoustic velocity $\mathbf{v}$ and acoustic pressure p (the time regime is $e^{-i\omega t}$, $\omega \in \mathbb{R}^+$) which are described by the coupled system in the framework of Landau and Lifshitz [2]

$$-i\omega\mathbf{v} + \nabla p - R^{-1}\Delta_\eta \mathbf{v} = \mathbf{f}, \text{ in } \Omega, \tag{1a}$$

$$-i\omega p + \operatorname{div}\mathbf{v} = 0, \text{ in } \Omega, \tag{1b}$$

$$\mathbf{v} = \mathbf{0}, \text{ on } \partial\Omega. \tag{1c}$$

In the *momentum equation* (1a) with some known source term $\mathbf{f}$ the viscous dissipation in the momentum is not neglected as we consider near wall regions. Here, $R^{-1} = \eta/(\rho_0 c L) \ll 1$ is a dimensionless number, c the sound velocity, L the characteristic length of the domain, and $\Delta_\eta := \Delta + (\frac{1}{3} + \zeta/\eta)\nabla \operatorname{div}$ with $\zeta \geq 0$ the second (volume) viscosity. The *continuity equation* (1b) relates the acoustic pressure linearly to the divergence of the acoustic velocity. The system is completed by *no-slip* boundary conditions (1c). Here we assume that $\mathbf{f} = 0$ on $\partial\Omega$, more general results can be found in [1].

Asymptotic expansion

The acoustic equations (1) show a viscosity boundary layer of thickness $O(\sqrt{R^{-1}})$ for the tangential component of the velocity. Introducing the small parameter $\varepsilon = \sqrt{R^{-1}}$ and curvilinear coordinates (t, s) close to the boundary where t is the tangential variable and s the normal one, we write the solution of (1) inspired by the framework of Vishik and Lyusternik [3] as

$$\mathbf{v} = \sum_{j=0}^{\infty} \varepsilon^j \left(\mathbf{v}^j + \varepsilon\, \mathbf{curl}_{2D}\, \phi^j\right); \; p = \sum_{j=0}^{\infty} \varepsilon^j p^j, \tag{2}$$

where $\mathbf{v}^j(x, y)$ and $p^j(x, y)$ are terms of the *far field* expansion, the *near field* terms $\phi^j(t, \frac{s}{\varepsilon})$ represent the boundary layer close to the wall, and $\mathbf{curl}_{2D} = (\partial_y, -\partial_x)^\top$.

The method of multiscale expansion separates the far and near field terms. The far field velocity terms $\mathbf{v}^j$ satisfy the partial differential equation (PDE)

$$\nabla \operatorname{div}\mathbf{v}^j + \omega^2\mathbf{v}^j = i\omega\mathbf{f}\cdot\delta_{j=0} + i\omega\Delta_\eta\mathbf{v}^{j-2}, \tag{3a}$$

$$\mathbf{v}^j(t, 0)\cdot\mathbf{n} = \partial_t\phi^{j-1}(t, 0), \tag{3b}$$

where $\phi^{-j} \equiv 0$ for $j < 0$, $\delta_{j=0}$ the Kronecker symbol which is 1 if $j = 0$ and 0 otherwise, and $\mathbf{n}$ the outer normal vector. The far field pressure terms follow as

$$p^j = -\frac{i}{\omega}\operatorname{div}\mathbf{v}^j. \tag{4}$$

The near field terms $\phi^j(t, S)$ for $S \in [0, \infty)$ are defined by the ordinary differential equation (ODE)

$$\begin{aligned} i\omega\phi^j + \partial_S^2\phi^j &= \kappa\big(3\,i\omega S + 3S\partial_S^2 + \partial_S\big)\phi^{j-1} - \partial_t^2\phi^{j-2} \\ &+ \big(-3\,i\omega\kappa^2 S^2 - 3\kappa^2 S^2\partial_S^2 - 2\kappa^2 S\partial_S\big)\phi^{j-2} \\ &+ \big(i\omega\kappa^3 S^3 + \kappa^3 S^3\partial_S^2 + \kappa^3 S^2\partial_S + \kappa S\partial_t^2 - \kappa' S\partial_t\big)\phi^{j-3}, \end{aligned} \tag{5a}$$

with the boundary condition

$$\partial_S\phi^j(t, 0) = \mathbf{v}^j(t, 0)\cdot\mathbf{n}^\perp, \tag{5b}$$

and decay condition for $S \to \infty$. Here, $\mathbf{n}^\perp = (n_2, -n_1)^\top$ and κ are tangential vector and curvature on $\partial\Omega$. Note, that in this ODE the tangential variable t is just a parameter.

The far field velocity term $\mathbf{v}^0$ has only a vanishing normal component, and the tangential component gets zero only if $\varepsilon\,\mathbf{curl}_{2D}\,\phi^0(t, \frac{s}{\varepsilon})$ is added, see (1c), where the zeroth order near field function for $S = \frac{s}{\varepsilon}$

$$\phi^0\big(t, \frac{s}{\varepsilon}\big) = \frac{1-i}{\sqrt{2\omega}}\exp\big(-\frac{(1+i)}{\sqrt{2}}\frac{\sqrt{\omega}s}{\varepsilon}\big)\;\mathbf{v}^0(t, 0)\cdot\mathbf{n}^\perp$$

decays exponentially away from the boundary. The sum $\mathbf{v}^0 + \varepsilon\,\mathbf{curl}_{2D}\,\phi^0(t, \frac{s}{\varepsilon})$ has a non-zero, but small normal component and is therefore corrected by $\varepsilon\mathbf{v}^1$. The far and near fields are iteratively computed as follows:

$$\begin{array}{ccccccccccc} \mathbf{f} & \xrightarrow{\text{PDE (3)}} & \mathbf{v}^0 & \xrightarrow{\text{ODE (5)}} & \phi^0 & \xrightarrow{\text{PDE (3)}} & \mathbf{v}^1 & \xrightarrow{\text{ODE (5)}} & \phi^1 & \xrightarrow{\text{PDE (3)}} & \ldots \\ & & \downarrow{\scriptstyle (4)} & & & & \downarrow{\scriptstyle (4)} & & & & \\ & & p^0 & & & & p^1 & & & & \end{array}$$

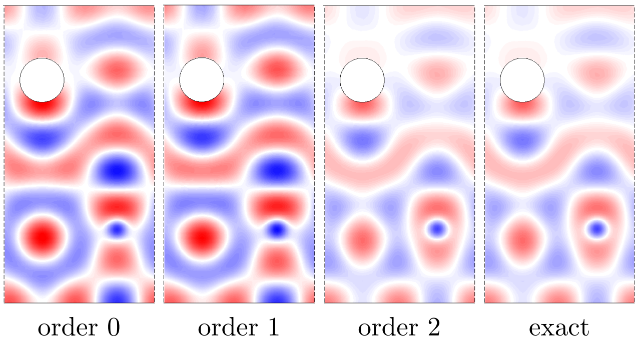

Figure 1: Comparison of the real part of the pressure.

Impedance boundary conditions

Outside a $O(\varepsilon)$-neighbourhood of the boundary the far field velocity $\sum_{j=0}^{N} \varepsilon^j \mathbf{v}^j$ serves as accurate approximation to $\mathbf{v}$, where the error is the smaller the higher N. With approximative models and impedance boundary conditions approximations $\mathbf{v}_{\text{appr},N} \approx \mathbf{v}^{\varepsilon,N}$ shall be defined by a single PDE, respectively, using again $R^{-\frac{1}{2}}$ instead of ε.

Impedance boundary conditions for the velocity. We derived the approximative models for $N = 0$,

$$\nabla \operatorname{div} \mathbf{v}_{\text{appr},0} + \omega^2 \mathbf{v}_{\text{appr},0} = \mathrm{i}\omega \mathbf{f}, \tag{6a}$$

$$\mathbf{v}_{\text{appr},0} \cdot \mathbf{n} = 0, \tag{6b}$$

for $N = 1$

$$\nabla \operatorname{div} \mathbf{v}_{\text{appr},1} + \omega^2 \mathbf{v}_{\text{appr},1} = \mathrm{i}\omega \mathbf{f}, \tag{7a}$$

$$\mathbf{v}_{\text{appr},1} \cdot \mathbf{n} - \frac{1}{\omega^2} \frac{(1+\mathrm{i})}{\sqrt{2\omega R}} \partial_t^2 \operatorname{div} \mathbf{v}_{\text{appr},1} = 0, \tag{7b}$$

and for $N = 2$

$$(\nabla \operatorname{div} - \tfrac{\mathrm{i}\omega}{R} \Delta_\eta) \mathbf{v}_{\text{appr},2} + \omega^2 \mathbf{v}_{\text{appr},2} = \mathrm{i}\omega \mathbf{f}, \tag{8a}$$

$$\mathbf{v}_{\text{appr},2} \cdot \mathbf{n} - \frac{1}{\omega^2} \Big(\frac{(1+\mathrm{i})}{\sqrt{2\omega R}} \partial_t^2 \operatorname{div} \mathbf{v}_{\text{appr},2} + \frac{\mathrm{i}}{2\omega R} \partial_t (\kappa \partial_t \operatorname{div} \mathbf{v}_{\text{appr},2}) \Big) = 0. \tag{8b}$$

Compare also the impedance conditions of 1^{st} order in [5].

Impedance boundary conditions for the pressure. For any N the quantity $p_{\text{appr},N} := \frac{1}{\mathrm{i}\omega} \operatorname{div} \mathbf{v}_{\text{appr},N}$ shall approximate p accurately even up to the boundary, as the pressure does not show boundary layer behaviour. Applying div to (6a), (7a) or (8a), respectively, and using the identity $\operatorname{div} \mathbf{curl}_{2D} \equiv 0$ we observe Helmholtz equations for $p_{\text{appr},N}$, and we get impedance boundary conditions after evaluating the normal component of (1a) and using (6b), (7b) and (8b). The resulting approximative models are for $N = 0$

$$\Delta p_{\text{appr},0} + \omega^2 p_{\text{appr},0} = \operatorname{div} \mathbf{f},$$
$$\nabla p_{\text{appr},0} \cdot \mathbf{n} = 0,$$

for $N = 1$

$$\Delta p_{\text{appr},1} + \omega^2 p_{\text{appr},1} = \operatorname{div} \mathbf{f},$$
$$\nabla p_{\text{appr},1} \cdot \mathbf{n} + \frac{1+\mathrm{i}}{\sqrt{2\omega R}} \partial_t^2 p_{\text{appr},1} = 0,$$

and for $N = 2$

$$\Big(1 - (\tfrac{4}{3} + \tfrac{\zeta}{\eta}) \tfrac{\mathrm{i}\omega}{R}\Big) \Delta p_{\text{appr},2} + \omega^2 p_{\text{appr},2} = \operatorname{div} \mathbf{f},$$
$$\nabla p_{\text{appr},2} \cdot \mathbf{n} + \frac{1+\mathrm{i}}{\sqrt{2\omega R}} \partial_t^2 p_{\text{appr},2} + \frac{\mathrm{i}}{2\omega R} \partial_t (\kappa \partial_t p_{\text{appr},2}) = 0.$$

Error estimate

Let us state an error estimate for the previously defined velocity and pressure approximations.

Lemma If ω^2 is not a Neumann eigenvalue of $-\Delta$, then, for any $\delta > 0$ and Ω_δ the domain Ω without a δ-neighbourhood of $\partial\Omega$ there exists a constant C_δ and such that for $N = 0, 1, 2$

$$\left. \begin{array}{r} \| \mathbf{v} - \mathbf{v}_{\text{appr},N} \|_{H(\text{div},\Omega_\delta)} \\ R^{-\frac{1}{4}} \| \operatorname{curl}_{2D} (\mathbf{v} - \mathbf{v}_{\text{appr},N}) \|_{L^2(\Omega_\delta)} \\ \| p - p_{\text{appr},N} \|_{H^1(\Omega)} \end{array} \right\} \le C_\delta R^{-\frac{N+1}{2}}.$$

For a rectangular domain with omitted disk we have performed numerical simulations for the exact model (1) and the approximative pressure models, see Fig. 1. We have used high-order finite elements within the numerical C++ library *Concepts* [4] to push the discretisation error below the modelling error. Figure 2 shows the modelling error in dependance of the viscosity.

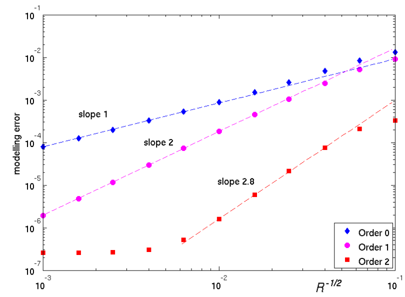

Figure 2: The modelling error $\|p - p_{\text{appr},N}\|_{H^1(\Omega)} + \|\mathbf{v} - \mathbf{v}_{\text{appr},N}\|_{H(\text{div},\Omega)}$ for $N = 0, 1, 2$ w.r.t. viscosity.

Conclusion

With the technique of multiscale expansion we have derived approximative models up to order 2 approximating the velocity and pressure for small viscosities. These are partial differential equations and (generalised) impedance boundary conditions which relate on the boundary the normal component of the velocity to its divergence or the normal derivative of the pressure to the pressure itself.

References

[1] Schmidt, K. and Thöns-Zueva, A. Preprint 2012-4, Inst. f. Math., TU Berlin, 2012

[2] Landau, L. D. and Lifshitz, E. M.: Fluid Mechanics. Course of theoretical physics, Vol. 6. Pergamon press, New York, 1959.

[3] Vishik, M.I. and Lyusternik, L.A. Russian Math. Surveys, 1960, 15:4, pp. 23–91.

[4] Webpage of Numerical C++ library *Concepts*, `http://www.concepts.math.ethz.ch`, 2012.

[5] Aurégan, Y., Starobinski, R. and Pagneux, V. Int. J. Aeroacoustics, 2001, 109:1, pp. 59–64.