A Rigorous Definition of the Term "Diffuse Sound Field" – and why the Sabine Reverberation Formula is Different from the Eyring Formula

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Introduction

The term "diffuse sound field" **(DSF)** is often not explained very accurately. In this rather didactical paper a more rigorous definition is proposed and the relationships to the necessary surface conditions as absorption and scattering are discussed. First, the general condition of geometric/statistic room acoustics is presupposed that typical room dimensions are large compared with wavelengths such that the analysis may be performed with an energetic sound particle model. So, 'Intensity' I is here interpreted as an integral over the whole solid angle: $I = \int j d\Omega$, a scalar rather than a vector:

$I = c\,U$ (c= sound velocity, U= energy density) (1).

The Eyring and Sabine reverberation formulae will be re-derived beside the mean free path length formula and the reason for the difference between both formulae will be analysed. Also a transition model is proposed.

Conditions for the diffuse sound field (Table 1)

First, it should be distinguished between

theoretical definitions (left) and practical conditions (right), further between the claim the sound field should be diffuse 'from the start' (strict version) or 'towards the end of reverberation" (tolerant version, conditions in brackets):

A) Isotropy (j=const)	**C)** all absorption degrees zero ! (mean absorption degree small)
B) Homogeneity (U=const)	**D)** all surfaces totally scattering i.e Lambert diffuse reflections (only some surfaces scattering)
B2) constant irradiation of all surfaces	**D2)** totally mixing

Table 1

Usually one starts with A (' **each direction with same intensity /** 'directional diffusivity').

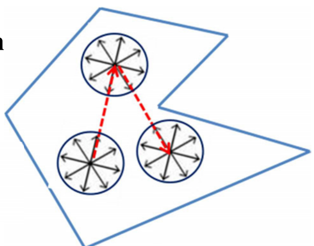

Fig. 1: Isotropy and homogeneity (same arrow lengths in every direction everywhere)

From A follows B (in a room without absorption, the particles don't lose energies, see the lines connecting the clusters in fig. 1) [1], but not vice versa (consider e.g. the case of a long room evenly filled with rays just in a longitudinal direction). From B (volume condition) follows B2. The surface conditions C+D are necessary but not sufficient for A+B!(All walls may be totally diffusely reflecting, but the irradiation strengths may be non-constant due to geometry.) If just one piece of surface is absorbing or specularly reflecting (producing a mirror source) then the sound field closely in front will be not isotropic.

With the Lambert reflection, the reflection angle (ϑ) probability density p' per solid angle is proportional $cos(\vartheta)$, independent from the incidence angle:

$$p' =: \frac{dp}{d\Omega} = \frac{cos(\vartheta)}{\pi} \qquad (2)$$

It can be considered as the ideal scattering characteristics following from the projection law onto rough surfaces.

If one is content with just a convergence towards a diffuse sound field after many reflections, then a room just with some scattering surfaces , hence, a little bit mixing is enough (and for the Sabine formula the condition that just the average absorption degree is ' small'). Only if (fictively) the surface were also interchanging positions (evenly distributed), i.e. totally mixing (D2), then from C+D+D2 follows A+B +B2 i.e. constant irradiation strength $B = I/4$ (3)

A constant B is the condition that the notion 'equivalent absorption area' (used to derive the Sabine formula) i.e. a surface-weighting, makes sense. The factor 1/4 is due to the directional averaging over the projection factor $cos(\vartheta)$.

Average quantities in a DSF and conditions

The core physical quantity is the equivalent absorption area

$$A = \sum \alpha_i S_i \qquad (4)$$

or the 'mean absorption degree' $\alpha_m \equiv \alpha = \frac{\sum \alpha_i S_i}{S} = A/S$ (5)

(S_i= single surfaces, S= surface, α_i = absorption degrees).

The other, the geometric average quantity, is the mean free path length (mfp) with its famous formula $\Lambda = 4V/S$ (6) (V=volume), which is true even for non-convex rooms, ***if*** a diffuse sound field really were given – which is, however, hardly the case then. The same is valid for the other relationships. The correct mfp- formula can be derived strictly obeying conditions A…D2 which shall be explicitly named here:

Method a) is utilizing

A) isotropy in *Ω* + B) homogeneity in *V* + averaging over the **inverse** mfp, i.e. reflection frequencies ('time average'):

$$\Lambda^{-1} = \overline{l^{-1}}^{V,\Omega} \qquad (7)$$

This way is gone with the derivation of the Sabine formula. In a DSF, 'sound particles' lose their identity: 'time = ensemble average' [1]. Therefore the averaging can be performed also over a group of 'parallel' rays into an absolute direction δ representative for all (fig.2).

$$\Lambda(\delta) = \frac{1}{n}\sum_{i=1}^{n} l_i = \frac{V/q}{Q/q} = \frac{V}{Q(\delta)} \quad (8)$$

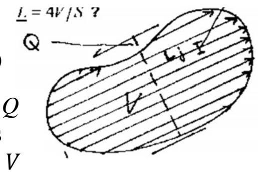

Fig. 2: A volume V of cross section Q subdivided into n 'channels' of cross section q and lengths l_i: $q\sum_{i=1}^{n} l_i = V$

The inverse of eq. 8 inserted in eq. 7 yields

$$\frac{1}{\Lambda} = \overline{\left(\frac{1}{\Lambda(\delta)}\right)}^{\Omega} = \overline{\left(\frac{Q(\delta)}{V}\right)}^{\Omega} = \frac{S}{4V} \qquad (9)$$

as the directional average over all projected surface elements dS (also over their backsides) is $dS/4$. (The average cross section of any volume is always $Q=S/4$).

Method b) is utilizing B2) constant irradiation of *S* + D) everywhere Lambert law + direct averaging over the mfp ('ensemble average'): $\boldsymbol{\Lambda} = \overline{l\,p'}^{\Omega,S}$ (10)

$$\boldsymbol{\Lambda} = \frac{1}{S}\int_S \int_{2\pi} l(\vartheta)\frac{cos(\vartheta)}{\pi}\, d\Omega dS \qquad (11)$$

ϑ is the local angle relative to the normal. The surface integral is independent from orientation 2V and can be separated: $\boldsymbol{\Lambda} = \frac{1}{\pi S}\int_{2\pi} d\Omega \int_S l(\vartheta)cos(\vartheta)dS = \frac{2\pi\, 2V}{\pi S} = \frac{4V}{S}$ (12)

Re-derivation of the Eyring formula

Typical is here to consider a ´representative sound particle` (sp) which, after always a free path length Λ, 'sees' a surface with the absorption degree α_m. The consequence is a step-wise exponential energy decay:

$$E(N) = E_0(1-\alpha_m)^N \qquad (13)$$

where E_0 is the start energy and N is the reflection number. By introducing a mean absorption exponent

$$\alpha'_m = -\ln(1-\alpha_m) \qquad (14)$$

equ. 13 reads $E(N) = E_0 e^{-N\alpha_m'}$. For N reflections with a free path length Λ, the time $t = N\,\Lambda/c$ is needed. Tacitly it is assumed that N is a real number as after a switched off steady sound source the decays overlap and 'smooth' the resulting function $E(t) = E_0 e^{-\alpha'_m ct/\Lambda}$. Using the standard formulation of an exponential decay $E(t) = E_0 e^{-\frac{t}{\tau}}$ the time constant of the sound energy decay is

$$\tau_{ey} = \frac{\Lambda}{c\,\alpha_m{'}} \qquad (15)$$

In such a typical RT formula, the time constant is always the proportion of the mean free path length and an average absorption exponent. The time for a 60dB decay is then generally $T = 6\ln(10)\,\tau$. Using the normalized value of the sound velocity c=340m/s at 14°C and also the value for the mfp Λ (equ. 6) yields the Eyring reverberation time

$$T_{ey} = \frac{6\ln(10)}{c}\frac{4V}{S\,\alpha_m{'}} \approx 0.163\,\frac{V}{S\,\alpha_m{'}} \qquad (16)$$

Additionally (to the DSF) is assumed: condition D2 (only with a total mixing the sp lose their identity) and: 'the mfp are constant' –which is of course wrong: they are varying.

Re-derivation of the Sabine formula

The Sabine formula is not just an approximation of the Eyring formula, it has its own, amazingly different derivation. Neither the model of a sp nor the concept of a mfp is used. Instead, the **decay of the total sound energy E(t)** (one value everywhere!) is considered aiming at a differential equation. With the energy density $U = E/V$, $I = c\,U$, the irradiation strength is $B = I/4 = cU/4 = cE/(4V)$ (see equ.3). Then the incident energy per time is

$$\frac{dE_i}{dt} = cE\,\frac{S}{(4V)} \qquad (17),$$

and finally the absorbed energy : $\frac{dE}{dt} = -E\,c\frac{\alpha_m S}{4V}$ (18).

The solution is an exponential decay with the time constant

$$\tau_{sab} = \frac{\Lambda}{c\,\alpha_m} \qquad (19)$$

Inserting again $\Lambda = 4V/S$ yields the famous Sabine RT

$$T_{sab} = 6\cdot\ln(10)\,\tau_{sab} \approx 0.163\,\frac{V}{A} \qquad (20).$$

As the energy is proportional the number of sound particles, analogously to equ. 17 the sp impact rate is

$$dN/dt = N/t = cN\,S/(4V) \qquad (21)$$

(constant without absorption). After the time for travelling just a mfp $t = \Lambda/c$, all sp once have hit the room surface. Hence, inserting t into equ. 21 yields by the way a prove for the formula for the mean free path length $\Lambda = 4V/S$.

Why is the Sabine different from the Eyring formula?

For small α_m, $\alpha'_m = -\ln(1-\alpha_m) \approx \alpha_m(1+\frac{\alpha_m}{2})$ (22), so comparing both formulae (equs. 15 and 19) shows that the difference is just in the order of $\frac{T_{ey}}{T_{sab}} \approx (1-\alpha_m/2)$ (23).

The reasons for the difference are the different tacit additional assumptions. The Sabine assumption of only one energy value is absurd as this were only possible if the information about absorption at one part of the surface were spread infinitely within the room.

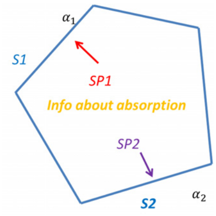

Fig.3: If SP1 hits wall S1,the total energy E in the room is reduced. Tacitly assumed by Sabine, this information is transmitted immediately to SP2 such that its energy is also reduced. Then its future energy loss due to absorption on wall S2 will be smaller than otherwise.

Hence, with the Sabine theory there is apparently a smaller effective energy decay than with Eyring.

Transitions between the two formulae

Starting with the Eyring model, obviously one has to consider the time interval between two reflections $\Delta t = \Lambda/c$ more in detail. The first thinking model is to subdivide it. With an equally distributed time shift of <u>many</u> sound particles, the energy loss may be linearly interpolated, so, after a time $\Delta t/n$, the overall energy loss factor is $(1-\alpha/n)$; allowing an 'information and energy inter change' the energy loss after '1/n reflection' would be 'equalized'. So, after a whole reflection – n such steps - the energy would be multiplied by $(1-\alpha/n)^n$. For $n \to \infty$, the loss factor between two reflections would become $f_{sab} = \lim_{n\to\infty}\left(1-\frac{\alpha}{n}\right)^n = e^{-\alpha}$ (24).

This is the Sabine energy loss factor for 1 reflection. An idea to describe the opposite transition is to assume that for the energy loss at the surface the energy (considered with the Sabine model) in the middle of the room is relevant. Thus the former diff. equ.18 with $\Lambda = 4V/S$ has to be altered to

$$\frac{dE}{dt} = -\frac{c\,\alpha}{\Lambda}E(t-\Delta t/2) \qquad (26)$$

where $\Delta t = \frac{\Lambda}{c} = \alpha\tau_{sab}$ is the half of the time interval between two reflections. This leads back to the Eyring RT.

It shall be mentioned that with allowing a variance of the free path lengths, one returns (for a maximum variance) from the Eyring RT towards the Sabine value, as Kuttruff showed [1]. Taking into account also a variance of the absorption degrees, the Kuttruff reverberation time is even smaller than the Eyring value [1]: $T_{sab} >\approx T_{Ey} >\approx T_{Kutt}$.

Conclusion

The "diffuse sound field" is a very idealistic assumption- The reasons for the difference between the Sabine and the Eyring formula are different tacit additional assumptions. Both are wrong. So, strictly speaking, they must not be applied in many cases of non-perfectly diffuse reflections i.e. in many realistic cases. The RT in non-diffuse sound fields depend on the room shape, the distribution of the absorption and especially the scattering coefficients σ [2].

References

[1] Kuttruff, H.: Room Acoustics, Elsevier Science Publishers Ltd., Barking, England, 3rd ed. (1991)

[2] Stephenson, U.M.: The Reverberation Time as a Function of the Surface Scattering Coefficients; in: Fortschritte der Akustik, DAGA 2012, Hrsg. DPG-GmbH, Bad Honnef, 2012