Spherical Slepian functions for approximation of spherical measurement data

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Introduction

Directivity patterns can only be measured at discrete directions. In many cases, the sound pressure between the measurement points is also of interest. Directivity patterns are assumed to be smooth, hence permit approximation with spherical harmonics of limited order. Spherical harmonic approximation is well-behaved if discrete measurements are available for sufficiently many, uniformly spaced directions. However, if the measurements cover only a limited range of directions, e.g. the half space, spherical harmonic approximation becomes inaccurate or ill-posed.

Spherical Slepian functions are useful for approximating measurement data that are available only for a limited range of directions. This contribution shows how to obtain suitable spherical Slepian functions from the spherical harmonics. Applying these functions yields accurate and well-posed approximation within the measured angular range. Such Slepian functions are also applicable in Ambisonics, cf. [2], directional source descriptions in spatial audio, and head related transfer functions.

Spherical harmonics expansion

Within this article, we define the Cartesian direction vector as $\boldsymbol{\theta} = [\cos(\varphi)\sin(\vartheta), \sin(\varphi)\sin(\vartheta), \cos(\vartheta)]^{\mathrm{T}}$ with φ and ϑ being the azimuth and zenith angle, respectively.

A continuous angularly band-limited function on the sphere $p(\boldsymbol{\theta})$ can be expressed by a series expansion,

$$p(\boldsymbol{\theta}) = \sum_{n=0}^{\mathrm{N}} \sum_{m=-n}^{n} Y_n^m(\boldsymbol{\theta})\, \psi_{nm}, \tag{1}$$

where ψ_{nm} are the expansion coefficients and

$$Y_n^m(\boldsymbol{\theta}) = N_n^{|m|} P_n^{|m|}(\cos\vartheta) \begin{cases} \cos(m\varphi)\,, & \text{for } m \geq 0 \\ \sin(m\varphi)\,, & \text{for } m < 0 \end{cases}$$

are the spherical harmonics (SHs) of order n and degree m; $P_n^{|m|}$ denotes the associated Legendre function, and $N_n^{|m|}$ is the scalar energy-normalization of the SHs. The double sum in Eq.(1) is equivalently expressed by a vector product

$$p(\boldsymbol{\theta}) = \boldsymbol{y}_{\mathrm{N}}^{\mathrm{T}}(\boldsymbol{\theta})\, \boldsymbol{\psi}_{\mathrm{N}}, \tag{2}$$

where $\boldsymbol{y}_{\mathrm{N}}(\boldsymbol{\theta}) := [Y_n^m(\boldsymbol{\theta})]_{q=1\ldots(\mathrm{N}+1)^2}$ and $\boldsymbol{\psi}_{\mathrm{N}} = [\psi_n^m]_{q=1\ldots(\mathrm{N}+1)^2}$ with the linear index $q = n^2+n+m+1$.

The first step for decomposition into SHs is done by multiplication of Eq.(2) with $\boldsymbol{y}_{\mathrm{N}}(\boldsymbol{\theta})$ and integration over the unit sphere $\mathbb{S}^2$

$$\int_{\mathbb{S}^2} \boldsymbol{y}_{\mathrm{N}}(\boldsymbol{\theta})\, p(\boldsymbol{\theta})\, \mathrm{d}\boldsymbol{\theta} = \boldsymbol{G}\, \boldsymbol{\psi}_{\mathrm{N}}, \tag{3}$$

where $\boldsymbol{G} = \int_{\mathbb{S}^2} \boldsymbol{y}_{\mathrm{N}}(\boldsymbol{\theta})\, \boldsymbol{y}_{\mathrm{N}}^{\mathrm{T}}(\boldsymbol{\theta})\, \mathrm{d}\boldsymbol{\theta}$. The second step is the inversion of $\boldsymbol{G}$. It is not required here as the SHs are orthonormal functions on the unit sphere, i.e. $\boldsymbol{G} = \boldsymbol{I}_{\mathrm{N}}$.

Discrete SHs expansion

Typically, measurement data is only available at L discrete directions. Therefore it is necessary to formulate a discrete SHs expansion. We define a vector $\boldsymbol{p} := [p(\boldsymbol{\theta}_l)]_{l=1\ldots\mathrm{L}}$ containing the measured values and a matrix $\boldsymbol{Y}_{\mathrm{N}} := [\boldsymbol{y}_{\mathrm{N}}(\boldsymbol{\theta}_l)]^{l=1\ldots\mathrm{L}}$ containing the angularly sampled SHs at the measurement directions $\boldsymbol{\theta}_l$, and with that the discrete equivalent of Eq.(2) is expressed as

$$\boldsymbol{p} = \boldsymbol{Y}_{\mathrm{N}}\;\; \boldsymbol{\psi}_{\mathrm{N}}. \tag{4}$$

Similarly as above, the first step for decomposition is done by multiplication with $\boldsymbol{Y}_{\mathrm{N}}^{\mathrm{T}}$ from the left,

$$\boldsymbol{Y}_{\mathrm{N}}^{\mathrm{T}}\, \boldsymbol{p} = \boldsymbol{G}\, \boldsymbol{\psi}_{\mathrm{N}}, \tag{5}$$

with $\boldsymbol{G} = \boldsymbol{Y}_{\mathrm{N}}^{\mathrm{T}}\, \boldsymbol{Y}_{\mathrm{N}}$ Obviously, the second decomposition step requires the inverse of $\boldsymbol{G}$ to exist and to be numerically stable. Normally, this is no problem for measurement positions that are uniformly distributed, cf. [3], and sufficiently many $\mathrm{L} \geq (\mathrm{N}+1)^2$ to enable a full rank.

However, if the measurements cover only a limited range of directions, e.g. the half space, the inversion of $\boldsymbol{G}$ is numerically rank-deficient. As shown in the following, this defect already exists when decomposing continuous functions on a part of the sphere.

SHs expansion on parts of the sphere

The first step of decomposing a continuous function $p(\boldsymbol{\theta})$ on a limited range of directions $S^2 \subset \mathbb{S}^2$ is done by

$$\int_{S^2\subset\mathbb{S}^2} p(\boldsymbol{\theta})\, \boldsymbol{y}_{\mathrm{N}}(\boldsymbol{\theta})\, \mathrm{d}\boldsymbol{\theta} = \boldsymbol{G}\, \boldsymbol{\psi}_{\mathrm{N}}, \tag{6}$$

with $\boldsymbol{G} = \int_{S^2\subset\mathbb{S}^2} \boldsymbol{y}_{\mathrm{N}}(\boldsymbol{\theta})\, \boldsymbol{y}_{\mathrm{N}}^{\mathrm{T}}(\boldsymbol{\theta})\, \mathrm{d}\boldsymbol{\theta} \neq \boldsymbol{I}_{\mathrm{N}}$. In general, the inversion of $\boldsymbol{G}$ is numerically rank-deficient. This becomes apparent by eigendecomposition:

$$\boldsymbol{G} = \boldsymbol{U}\; \mathrm{diag}\{[\sigma_i]_{1\ldots(\mathrm{N}+1)^2}\}\; \boldsymbol{U}^{\mathrm{T}}, \tag{7}$$

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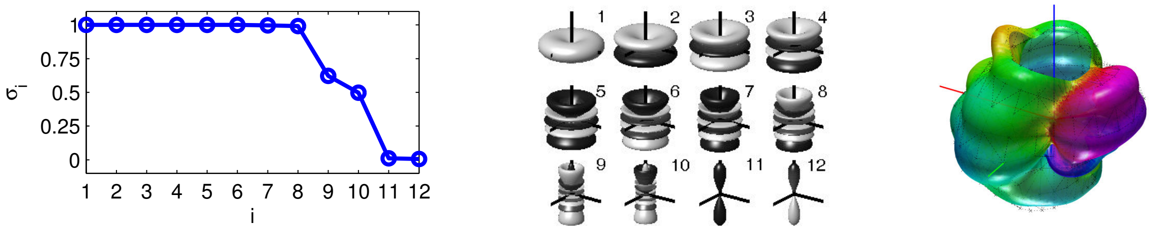

(a) **(b)** **(c)**

Figure 1: (a) shows eigenvalues according to Eq.(7) for SHs of $m = 0$ and $n \leq 11$ for ϑ limited within $[22.5°, 157.5°]$, (b) depicts the corresponding basis functions according to Eq.(8) of which 1–9 were chosen as Slepian functions, and (c) is the approximated directivity using such functions for all m, cf.[1].

where $\boldsymbol{U}$ is an orthogonal matrix containing the eigenvectors and σ_i are the corresponding eigenvalues. The eigenvalues of $\boldsymbol{G}$ decay gradually to zero as exemplarily sketched in Fig.1(a).

Spherical Slepian functions

Applied on the SHs, the eigenvector matrix $\boldsymbol{U}$ yields an orthogonal set of basis functions on S^2

$$\boldsymbol{v}_{\mathrm{N}}(\boldsymbol{\theta}) = \boldsymbol{U}\, \boldsymbol{y}_{\mathrm{N}}(\boldsymbol{\theta}) \tag{8}$$

that allows for an expansion of $p(\boldsymbol{\theta})$ that is equivalent to its SHs expansion

$$p(\boldsymbol{\theta}) = \boldsymbol{v}_{\mathrm{N}}^{\mathrm{T}}(\boldsymbol{\theta})\, \boldsymbol{\nu}_{\mathrm{N}}, \tag{9}$$

where $\boldsymbol{\nu}_{\mathrm{N}} = \boldsymbol{U}\, \boldsymbol{\psi}_{\mathrm{N}}$ are the new expansion coefficients. The first step of decomposition yields then

$$\int_{S^2 \subset \mathbb{S}^2} p(\boldsymbol{\theta})\, \boldsymbol{v}_{\mathrm{N}}(\boldsymbol{\theta})\, \mathrm{d}\boldsymbol{\theta} = \boldsymbol{G}\, \boldsymbol{\nu}_{\mathrm{N}}, \tag{10}$$

with a diagonal matrix $\boldsymbol{G} = \int_{S^2} \boldsymbol{v}_{\mathrm{N}}(\boldsymbol{\theta})\, \boldsymbol{v}_{\mathrm{N}}^{\mathrm{T}}(\boldsymbol{\theta})\, \mathrm{d}\boldsymbol{\theta} = \mathrm{diag}\{[\sigma_i]_{i=1\dots(\mathrm{N}+1)^2}\}$. Each of the basis functions $v_i(\boldsymbol{\theta})$ exhibits a concentration on S^2 that is proportional to its singular value σ_i.

A selection of functions $\tilde{\boldsymbol{v}}_{\mathrm{N}}(\boldsymbol{\theta}) = [v_i(\boldsymbol{\theta})]_{i=1\dots\mathrm{M}}$ that are sufficiently concentrated on S^2 defines the Slepian functions, cf. [4]. A suitable constant C is used to separate the Slepian functions from the functions concentrated outside S^2, by the condition $\sigma_i > \mathrm{C}$. The decomposition into Slepian functions is then numerically stable

$$\tilde{\boldsymbol{\nu}}_{\mathrm{N}} = \mathrm{diag}\left\{\left[\frac{1}{\sigma_i}\right]_{i=1\dots\mathrm{M}}\right\} \int_{S^2 \subset \mathbb{S}^2} p(\boldsymbol{\theta})\, \tilde{\boldsymbol{v}}_{\mathrm{N}}(\boldsymbol{\theta})\, \mathrm{d}\boldsymbol{\theta}. \tag{11}$$

The expansion of discrete measurement data into Slepian functions is done similarly as in Eqs.(4) and (5) but using the Slepian functions instead of the SHs. In doing so, the matrix $\boldsymbol{G} = \tilde{\boldsymbol{V}}_{\mathrm{N}}^{\mathrm{T}}\, \tilde{\boldsymbol{V}}_{\mathrm{N}}$ appearing in Eq.(5) becomes regular for a uniformly discretized S^2.

Example

Slepian functions were applied to approximate directivity measurements of a super cardioid microphone with the influence of a talker's head, cf. [1]. The directivity was measured in 15° steps, in both azimuth and zenith angle. This allows for SHs approximation up to $\mathrm{N} = 11$. However, the measurement grid covered zenith angles from 30° to 150°, only, i.e. data on the polar caps is missing.

A Slepian basis can be found for each degree m individually because only the zenith angle is restricted. Fig.1(a) shows the eigenvalues σ_i corresponding to spherical Slepian functions for $m = 0$ restricting the zenith angle according to the measurement gird. Fig.1(b) shows the associated basis functions $\boldsymbol{v}_{\mathrm{N}}$. The basis functions with eigenvalues close to 1 are suitable for decomposition. In [1] we used the Slepian functions with $\sigma_i > 0.5$ for decomposition, i.e. the functions 1–9 in Fig.1(b) for $m = 0$. The functions for the other degrees are constructed similarly. Fig.1(c) exemplarily demonstrates the approximation of a measured directivity pattern.

Conclusion

A uniform distribution of measurement points on a limited angular region, e.g. the half space, does not allow a numerically stable decomposition into SHs. This defect also appears for the continuous SHs if the range of directions is limited, as they become linearly dependent. Re-orthogonalization of the SHs allows to remove the linear dependencies. The Slepian functions are defined by omitting those orthogonalized SHs that are negligible in the region of interest. This provides a continuous approximation tailored to the range of directions covered by the measurement grid.

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