Uncertainty Quantification of Stochastic Linear Systems under Random Impulse Loadings

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Abstract

Effects of uncertainty in system parameters and impulse loading characteristics on the response of single degree of freedom (SDOF) systems are studied. The uncertain system and loading parameters are represented by truncated generalized polynomial (gPC) expansions. The non-sampling stochastic simulation procedure based on the gPC expansion technique is used for dynamic analysis of the SDOF system duly considering the uncertainties. For a set of Hermite's collocation points, the system is analyzed to obtain response time histories and shock spectra. The statistical properties of response quantities are examined, and effectiveness of the gPC expansion based simulation procedureis compared with direct Monte Carlo (MC) simulation. Herein, five different impulse loads, applied on different SDOF systems, are considered. It is observed that shock spectra of all the pulses are sensitive to the uncertainties in region of time period close to pulse duration. Among other response quantities, acceleration is influenced more than the velocity and the displacement. The gPC expansion based simulation technique is observed to be an efficient alternative to computationally demanding MC simulation for quantifying uncertainties. The probability distributions for various response quantities, as obtained through the gPC expansion based simulations, are well compared with that obtained through direct MC simulations.

Introduction

In dynamics of structures, uncertainty is derived from a number of sources including stiffness, estimation of loads, deviations in dimensions, etc. Today, it is of paramount importance to consider the effects of these uncertainties on the output and perform probabilistic analysis of structures. A number of schemes have been developed in the past few decadesto quantify these effects. The underlying question in quantifying uncertainty is the method to be used to characterize the output of a physical process in which some of the parameters are stochastic processes and the solution is identified by its projection on certain basis functions chosen appropriately [1]. Number of techniques have evolved to model random processes and variables, such as Monte Carlo(MC) simulation, Karhunen–Loéve (KL) expansion, perturbation techniques, etc. One of the techniques to expand the random process over Hilbert space is generalized polynomial chaos, proposed in the first half of the last century [2, 3]. Ghanem and Spanos [4] used this technique in stochastic finite element analysis. Since then, gPC expansion technique has been used in a wide range of engineering applications.

It has been established by several researchers that gPC expansion approach is superior interms of computational efforts than MC simulations besides being faster converging [5, 6]. Kundu and Adhikari [7] used a stochastic Krylov subspace projection for investigating transient response of a randomly parameterized structural dynamic system and used gPC, with a deterministic impulse load. Manohar and Ibrahim [8] presented a brief review on the consideration of the parametric uncertainties in dynamic analysis ofstructures. Li and Chen [9] proposed an approach for dynamic responseanalysis of structures considering parameteric uncertainty using MC simulation considering uncertaintiesin structural parameters along with the random input excitation. It is imperative to run a large number of MC realizations for a reasonable accuracy in thesimulation, which may be extremely expensive in terms of simulation time. Baylot et al [10] studied uncertainties in blast loads using convention statistical procedures. Borenstein and Benaroya [11] performed sensitivity analysis of uncertain blast loading parameters on a clamped aluminum plate using MC simulations.

In the present work, single degree of freedom (SDOF) systems with stochastic material behaviour have been considered for studying the effects of uncertainties in various parameters including excitation characteristics on response quantities namely displacement, velocity and acceleration. Also, the probability distributions of response for different impulse loads are plotted and compared with those obtained using MC simulation. Five different impulse loadsare considered for illustration in the present study. The material uncertainty is considered through stiffness of SDOF system, and load uncertainty through impulse durationand its amplitude. In order to study the sensitivity towards uncertainties gPC approach has been used, expressing the parameters with truncated expansions.

Stochastic Reponse Modeling with gPC

Considering the uncertainties in stiffness k, and in impulse force f due to its uncertain amplitude F_0 and duration t_d, the equation of dynamic motion is written as

$$m\ddot{y}(t,\xi) + k(\xi_k)y(t,\xi) = f(t,\xi_f) \tag{1}$$

in which m is mass of the system; $y(t,\xi)$ is the unknown random displacement; ξ_k and ξ_f are the random variables identifying the uncertainties in stiffness and impulse load, respectively. It can be noted here that for impact loading, damping is ignored. Randomness in restoring force is inherent in that of the stiffness along with structural dimensions. The vector ξ represents all the random input vectors, and is given by $\xi = [\xi_k \, \xi_f]^{\mathrm{T}}$. The impulse load is expressed as

$$f\left(t,\xi_f\right) = F_0\left(\xi_{F_0}\right)\phi\left\{t, t_{\mathrm{d}}\left(\xi_{t_{\mathrm{d}}}\right)\right\} \tag{2}$$

where random variables ξ_{F_0} and $\xi_{t_{\mathrm{d}}}$ represent the uncertainties in amplitude and time duration of the impulse load and constitute ξ_f. Expression of the function ϕ depends upon the nature of the pulse. All the random variables belong to random Hilbert spaces and yield random vector ξ in the Hilbert space [12] and are assumed to be standard variables to preserve orthogonality properties in generalized polynomial expansion.

Uncertain structural parameter k, random loading f, and the impulse time duration t_d are expressed as truncated gPC expansion series [12,13] and have the following form, respectively.

$$k\left(\xi_k\right) = \sum_{n_1}^{N_1} \kappa_{n_1}\psi_{n_1}\left(\xi_k\right) \tag{3a}$$

$$F_0\left(\xi_{F_0}\right) = \sum_{n_2}^{N_2} \Phi_{0,n_2}\psi_{n_2}\left(\xi_{F_0}\right) \tag{3b}$$

$$t_{\mathrm{d}}\left(\xi_{t_{\mathrm{d}}}\right) = \sum_{n_3}^{N_3} \tau_{d,n_3}\psi_{n_3}\left(\xi_{t_{\mathrm{d}}}\right) \tag{3c}$$

where κ_{n_1}, Φ_{0,n_2} and τ_{d,n_3} are the deterministic unknown coefficients and $\psi_{n_1}\left(\xi_k\right)$, $\psi_{n_2}\left(\xi_{F_0}\right)$ and $\psi_{n_3}\left(\xi_{t_{\mathrm{d}}}\right)$ are the stochastic basis functions for stiffness, impulse load amplitude and its time duration, respectively. The random output $y\left(t,\xi\right)$ of the system is further represented as

$$y\left(t,\xi\right) = \sum_{n_4}^{N_4} \Upsilon_{n_4}\psi_{n_4}\left(\xi\right) \tag{4}$$

in which Υ_{n_4} is the deterministic coefficient vector and $\psi_{n_4}\left(\xi\right)$ is the stochastic basis function for the response. Substitution of Equations (2) to (4) into Equation (1) yields a stochastic error function as given below.

$$\begin{aligned} \varepsilon(t,\xi) = m\sum_{n_4}^{N_4} \ddot{\Upsilon}_{n_4}\psi_{n_4}\left(\xi\right) + k\left(\xi_k\right) + \sum_{n_1}^{N_1}\kappa_{n_1}\psi_{n_1}\left(\xi_k\right)\cdot\sum_{n_4}^{N_4}\Upsilon_{n_4}\psi_{n_4}\left(\xi\right) \\ -\sum_{n_2}^{N_2}\Phi_{0,n_2}\psi_{n_2}\left(\xi_{F_0}\right)\cdot\phi\left\{t, \sum_{n_3}^{N_3}\tau_{d,n_3}\psi_{n_3}\left(\xi_{t_{\mathrm{d}}}\right)\right\} \end{aligned} \tag{5}$$

The problem at hand consists of determining unknown deterministic coefficient vector Υ_{n_4} which is achieved by minimization of error function $\varepsilon(t,\xi)$ to zero, which starts from appropriately chosen stochastic basis functions. For Gaussian processes, these are chosen to be Hermite polynomials [12].

Determination of gPC Coefficients

In the present study, non-intrusive approach is used to minimize the error function. In this technique, the residual error is zero at specifically chosen collocation points over the random space Ω. Thus, Equation (5) becomes,

$$\int_\Omega \varepsilon(t,\xi)\,\delta\left(\xi_n - p_n^j\right)\rho(\xi_n)\,d\xi_n = 0, \quad \text{for } j = 1,2,\ldots,N_5 \tag{6}$$

where δ is the delta function, p_n^j is the set of collocation points chosen and $\rho(\xi_n)$ is probability distribution function (PDF) of n^{th} random variable ξ_n. The steps followed in the simulation of the problem here involve representing uncertain input parameters using truncated gPC expansions over a set of standard random variables ξ_i, where i denotes uncertain input parameter. The choice of generalized polynomial expansion depends upon the type of distribution of the random variable [12]. In the present study, the uncertain parameters are assumed normally distributed, thus any random process χ_i can be expressed by N^{th} order Hermite polynomial.

$$\chi_i\left(\xi_i\right) = \sum_{k=0}^{N} x_{i_k}\left\{H_k(\xi_i)\right\} \tag{7}$$

in which i refers to uncertain parameters k, F_0 or t_{d}, and x_{i_k} are the deterministic gPC coefficient which are obtained by Galerkin projection scheme [13] for input parameters.

(3a)

$$x_{i_k} = \frac{1}{\left\langle H_k^2\right\rangle}\int_{-\infty}^{\infty}\chi_i\left(\xi_i\right)H_l\left(\xi_i\right)\rho\left(\xi_i\right)d\xi_i, \quad \text{for } l = 0,1,\ldots,N \tag{8}$$

where $\left\langle H_k^2\right\rangle$ is the inner product in Hilbert space and given by, $\left\langle H_k^2\right\rangle = \int_{-1}^{1} L_k^2\left(\xi\right)\rho\left(\xi\right)d\xi$. The uncertain parameter χ_i being normally distributed is easily obtained in terms of its mean value and standard deviation as $\chi_i = \mu_i + \sigma_i\xi$, in which ξ is the standard normal variable.

The response of the system is represented with gPC expansion in a similar fashion and given as

$$Y_r\left(t,\xi\right) = \sum_{i=0}^{N_r} y_i(t)\psi_i\left(\xi\right) \tag{9}$$

where $y_i(t)$ is the unknown deterministic coefficient which varies with time, and $\psi_i\left(\xi\right)$ is the stochastic basis function as in Equation (4). The uncertain parameter in terms of Hermite polynomial H is given as [13],

$$\begin{aligned} \chi = x_0H_0 + \sum_{n_1=1}^{\infty}x_{n_1}H_1\left(\xi_{n_1}\right) + \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{n_1}x_{n_1n_2}H_2\left(\xi_{n_1},\xi_{n_2}\right) \\ +\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{n_1}\sum_{n_3=1}^{n_2}x_{n_1n_2n_3}H_3\left(\xi_{n_1},\xi_{n_2},\xi_{n_3}\right) + \ldots \end{aligned} \tag{10}$$

whichis substituted for stochastic basis function ψ in the equations above. As the number of terms in Equation (10) grows very fast with the number of uncertain parameters and the order of expansion, therefore, for optimum convergence and computational effort, the number of terms is truncated. To get the stochastic response of the system, the deterministic response is first computed at a specified number of collocation points. The minimum number of collocation points should be at least equal to the number of unknown coefficients in the gPC expansion series. These points are the roots of one order higher Hermite polynomial used,and zero. These points should be symmetrical about the origin. For one-dimensional random vectorξ and expansion up to third order, the response Y is written as

$$Y(t,\xi) = y_0(t)H_0 + y_1(t)H_1 + y_2(t)H_2 + y_3(t)H_3 \tag{11}$$

in which, $H_0 = 1$, $H_1 = \xi$, $H_2 = (\xi^2 - 1)$ and $H_3 = (\xi^3 - 3\xi)$; and $y(t)$'s are the time-varying deterministic gPC coefficients for each term in the expansion. The next step is computation ofthese coefficients using a set of deterministic outputs obtained at the selected collocation points. At every instant of time in the response history, a large set of simultaneous equations is solved using regression analysis based on least square method. These coefficients are substituted into the gPC expansions to obtain stochastic response time histories.

Stochastic Modeling for Shock Spectra

A similar gPC model is prepared for studying the effects of uncertainties on dynamic magnification factor (DMF) that is taken as the ratio of random peak displacement of system to deterministic static displacement. The dynamic magnification factor D_n for n^{th} SDOF system with time period T_n subjected to a given excitation pulse is obtained as

$$D_n(T_n,\xi) = \frac{\max \left| \sum_{i=0}^{N_t} y_i(t)\psi_i(\xi) \right|_{T_n}}{|y_{st}|} \tag{11}$$

The gPC expansion series for dynamic magnification factor D_n is given as

$$D_n(t,\xi) = \sum_{i=0}^{N_t} d_{n,i}(t) H_k(\xi_i) \tag{12}$$

Numerical Study

The above mentioned gPC formulation is used to analyse twenty SDOF systems of time periods ranging from 0.1s to 2s;with stochastic properties and subjected to five different types of random loads with time-histories as shown in Figure 1. The variation has been considered Gaussian. Mean values and standard deviations of each uncertain parameter are given in Table 1. The deterministic model of the system is solved for nine sets of uncertain input parameters. Time duration of response history is kept 2 seconds with time points after every 0.01s.

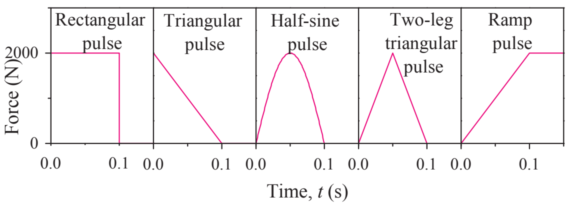

Figure 1: Impulse load definitions.

Table 1: Mean and standard deviation of the input parameters

Parameter	Mean (μ)	Standard deviation (σ)	Percetange Variation
t_d (s)	0.1	0.01	10
F_0 (N)	2000	400	20
k (N/m)	10000	1000	10
R_t (N)	20000	2000	10

The mean shock spectra obtained from the gPC expansion and compared with respective MC simulation and deterministic estimations (Figures 2). It is observed that for all the pulse loads, the mean stochastic shock spectra converge towards deterministic shock spectra for higher-time-period systems.

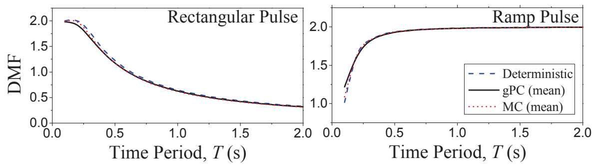

Figure 2: Comparison of mean shock spectra from gPC and MC with deterministic shock spectra for rectangular and ramp pulses.

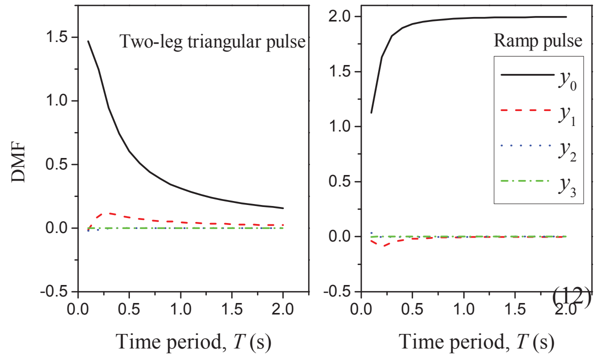

Figure 3: Variation of gPC coefficients for shock spectra for two-leg triangular and ramp pulses.

(13)

The gPC coefficients for shock spectra ordinates for two-leg triangular and ramp pulses are shown in Figure 3. The first coefficient (y_0) tracks the mean shock spectra, and higher order coefficients (y_1 and above) converge quite fast. The mean gPC and MC responses of SDOF system with $T = 0.1$ s are plotted in Figure 4, which are seen to be well in agreement with the respective determinsitic responses. For the same system, gPC coefficients are plotted for ramp pulse as shown in Figure 5. It is clear that the coefficient y_0 essentially tracks the mean responses and also dominates compared to higher order coefficients (y_1, y_2 and y_3). The coefficient y_1, which represents effect on variance of the response quantities, propagates uncertainty in the systems with lower time periods especially nearly equal to duration of the pulse (for $t_d/T \approx 1$). Third gPC coefficient (y_2) shows its effect on the systems with lower time period, as well as in later time histories of the responses. The acceleration is seen to be more sensitive towards uncertainties, followed by velocity and displacement since the third gPC coefficient (y_2) also shows up in some cases. It is further observed that, higher time period systems are less sensitive to uncertainties. Very small values of higher order gPC coefficients in all the cases indicate extent of convergence achieved with the gPC expansion formulated.

Furthermore, probability distribution functions (PDF) of DMFs at selected time periods are computed and compared with those given by 30,000 MC simulations. For illustration, Figure 6 shows PDF of velocity at selected time points for rectangular pulse applied on the system with time period 2 s. It is observed that the probability distributions of various response quantities including DMFs are fairly close to those obtained by MC simulations. It is observed that maximum probability densities of DMFs move closer to deterministic

DMFs for higher time period systems (for $t_d/T<<1$).The extent of occurrence of any response quantity is more or less symmetrical about deterministic values, and similar to that given by MC simulations.

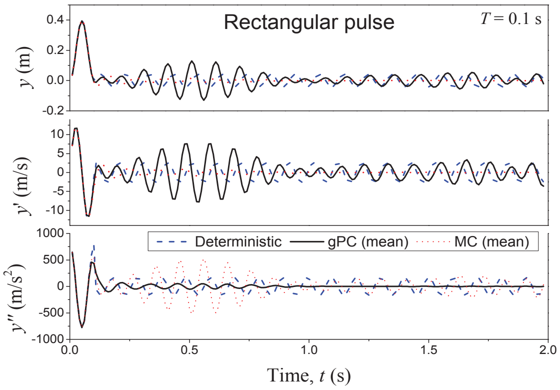

Figure 4: Mean gPC and MC responses of system with $T = 0.1$ s to rectangular pulse load.

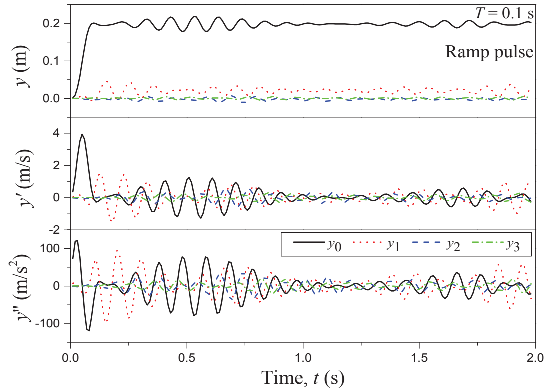

Figure 5: Variation of gPC coefficients of response of $T = 0.1$ s system for ramp pulse

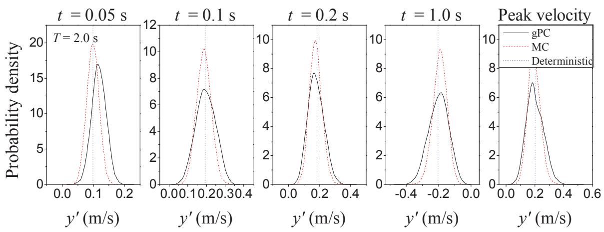

Figure 6: Probability distribution of velocity at selected time points for rectangular pulse for the three cases.

Conclusions

From the study presented here, following conclusions are drawn.

1. The gPC expansion qualifies to be a reasonably good substitute for MC simulations to quantify uncertainties and converges to an acceptable extent, for shock spectra as well as for other response quantities.
2. Shock spectra of impulse excitations are highly stochastic in the early region of time periods, especially when time period of the system is close to pulse duration.
3. Maximum probability density of response quantities including DMF, occurs close to deterministic values, more so for higher time periods.
4. Influence of uncertainty on response is more while impulse is acting. This is significant for the systems with time period equal to pulse duration.
5. Higher the order of response quantity more is the effect of uncertainties in input parameters. Accelerations are seen to be more sensitive towards randomness in inputs.

References

[1] Ghanem, R.:Propagation of uncertainty in mechanistic models of soil. Numerical Methods in Geotechnical Engineering (2000) 29-42.

[2] Wiener N.: The homogeneous chaos. American Journal of Mathematics, 60(4) (1938), 897-936.

[3] Cameron, R.H., Martin, W.T.:The orthogonal development of non-linear functionals in series of Fourier–Hermite functionals. Annals of Mathematics, 48 (1947), 385-392.

[4] Ghanem, R., Spanos,P.D.: Stochastic finite elements: aspectral approach. Springer-Verlag, New York, 1991.

[5] Sepahvand, K., Marburg, S., Hardtke, H.J.: Stochastic free vibration of orthotropic plates using generalized polynomial chaos expansion. Journal of Sound and Vibration331(1) (2012), 167-179.

[6] Brzakala, A.:Application of the polynomial chaos approximation to a stochastic parametric vibrations problem. Archives of Civil and Mechanical Engineering 13 (2013), 220-228.

[7] Kundu, A., Adhikari, S.:Transient response of structural dynamic systems with parametric uncertainty.Journal of Engineering Mechcanics140(2) (2014), 315-331.

[8] Manohar, C.S., Ibrahim, R.A.: Progress in structural dynamics with stochastic parameter variations. Applied Mechanics Review 52 (1999), 177-97.

[9] Li, J., Chen, J.B.: Probability density evolution method for dynamic response analysis of structures with uncertain parameters. Compt. Mech.34(5) (2004), 400-409.

[10] Baylot, J., Rickman, D.:Uncertainties in blast loads on structures. Proceedings of ASCEStructures Congress: New Horizons and Better Practices (2007), 1-10.

[11] Borenstein, E., Benaroya, H.: Sensitivity analysis of blast loading parameters and their trends as uncertainty increases, Journal of Sound and Vibration 321 (2009) 762-785.

[12] Sepahvand, K., Marburg, S.,Hardtke, H.J.: Uncertainty quantification in stochastic systems using polynomial chaos expansion. International Journal of Applied Mechanics 2(2)(2010), 305-353.

[13] Xiu, D., Karniadakis, G.: The Wiener Askey polynomial chaos for stochastic differential equations. SIAM Journal of Scientific Computing 24(2) (2002), 619-644.