Nitsche's method for Helmholtz boundary value problems and nonconforming finite element discretization

Polina Marinova, Otto von Estorff
Hamburg University of Technology, Institute of Modelling and Computation, 21073 Hamburg, Germany, Email: mub@tuhh.de

Introduction

The application of the finite element method for the approximation of sound transmission problems inevitably involves the imposition of the proper boundary conditions. While the natural boundary conditions can be directly incorporated in the variational formulation, the essential boundary conditions require often a special attention. The Lagrange multiplier and penalty methods are most frequently regarded approaches in this case, but along with the straight forward implementation they suffer from certain drawbacks, such as saddle-point structures, increased degrees of freedom, quite restrictive stability conditions, and distortion of the condition number of the resulting matrix problem. In this context a method proposed by Nitsche is quite appealing, since it imposes essential boundary conditions in a weak sense via boundary terms and maintains stability.

In the current contribution an implementation of the Nitsche's method for Dirichlet boundary conditions is presented and used to solve the Helmholtz equation with nonconforming discretization. The nonconforming finite element method is closely related to the discontinuous Galerkin method, which typically enforces weak continuity even on inter-element boundaries in a way originating from the Nitsche's work. Therefore a further implementation of the method for coupling non-matching meshes, in a case of nonconforming elements discretization is foreseen.[1]

Helmholtz equation

Sound transmission problems in the frequency domain involve the propagation of sound in fluids, which is described by the Helmholtz equation. In general this equation is written in terms of pressure and discretized with Lagrangian finite elements (FE), which in the case of multiple structure-fluid coupling causes non-symmetric eigenvalue problems. The other possibility is the displacement based formulation of the Helmholtz equation, whose treatment with Lagrangian FE leads to the appearance of nonzero spurious eigensolutions, which could be avoided with the use of finite elements designed to approximate properly vectorial unknowns as the Raviart-Thomas FE [6, 7]. In the current contribution the displacement based Helmholtz equation

$$\nabla(div\, u(x,y)) + k^2 p(x,y) = 0 \quad \text{in} \quad x, y \in [0,1] \qquad (1)$$

is taken and the related boundary value problem (BVP) is defined. A plane wave type excitation is determined as Neumann boundary condition (2),

$$\nabla u(x,y)\cdot n = -\frac{1}{i\rho c^2} p_n \quad \text{on} \quad \Gamma_N = 0\times[0,1] \qquad (2)$$

$$\nabla u(x,y)\cdot n - \frac{\omega}{ic^2\rho} A u = 0 \quad \text{on} \quad \Gamma_R = [0,1]\times 0,1 \qquad (3)$$

$$u(x,y)\cdot n = u_0 \quad \text{on} \quad \Gamma_D = 1\times[0,1] \qquad (4)$$

the absorbing properties are introduced in the form of Robin boundary condition (3) and a Dirichlet boundary condition (4) is assigned as a preparation for the coupling condition. For the unit square domain, considered here, the exact position, where each type of boundary conditions is applied, can be seen in Fig. 2. In the BVP formulation (1)-(4), k is the wave number, ω is the angular frequency, ρ_0 the fluid density, p_n the prescribed pressure and A the absorbing coefficient.

Mixed Raviart-Thomas formulation

In the mixed finite element method two different variables, in here displacement and pressure (u, p), are approximated by two spaces. The second variable is introduced, because of its physical interest and is related to the derivative of the original one. The mixed Raviart-Thomas finite element presented in this contribution uses the Raviart-Thomas space for the approximation of pressure unknowns, while the displacement unknowns are approximated with polynomials in a L^2 space [4].

Raviart-Thomas finite element

The simplest Raviart-Thomas elements is the lowest-order Raviart-Thomas element also called RT_0 [9]. On a triangle T the polynomial space for this element is defined by $RT_0 = [P_0(T)]^2 + [x_1, x_2]^T\, P_0(T)$, thus all vectors have the form

$$q = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + b \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \qquad (5)$$

The Raviart-Thomas element is designed to represent in particular the following Hilbert space

$$H(div,\Omega) = \left\{ q \in L^2(\Omega)^2 : div\; q \in L^2(\Omega) \right\}. \qquad (6)$$

The so-constructed space consist of all vectors with bounded divergence and the functions must have continuous normal components [10]. The lowest-order Raviart-Thomas element with indicated vertices, edges and edge normals is depicted in Fig. 1 along with the lowest-order Lagrangian element, in order to allow a comparison. Edge-basis functions with built-in continuity of the normal components on the boundaries are used,

$$\psi_{E_j} = \pm \frac{|E_j|}{2\,|T|}(x - P_j) \qquad j = 1, 2, 3 \quad \text{and} \quad x \in T. \quad (7)$$

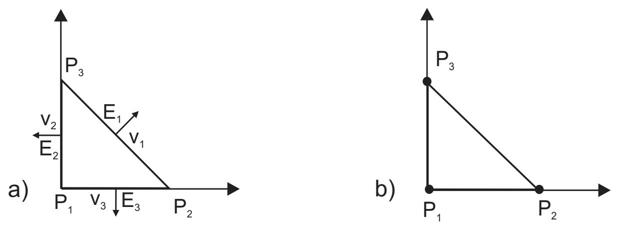

Figure 1: a) Lowest-order Raviart-Thomas element (RT_0) b) Lagrangian element

Mixed formulation for the displacement based Helmholtz equation

In order to get to the mixed formulation of the Helmholtz equation it is refered to the duality methods. A new unknown is introduce, which is also physically meaningful in this case, namely the pressure. The pressure p is associated to the derivatives of the displacement and the relation is expressed by the first equation in (8). Using this relation the original Helmholtz equation is reformulated and split into two equations, therefore the initial problem is transformed to a saddle point problem, where the saddle point (u, p) is searched for:

$$\begin{aligned} p(x, y) - c^2 \rho \nabla u(x, y) &= 0 \quad \text{in} \quad \Omega \\ \nabla p(x, y) - \omega^2 \rho u(x, y) &= 0. \end{aligned} \quad (8)$$

Special attention should be paid to function spaces for the dual mixed formulation,

$$\begin{aligned} H(div, \Omega) &:= \left\{ q \in L^2(\Omega)^2 : div\ q \in L^2(\Omega) \right\} \\ H_{0,N}(div, \Omega) &:= \{ q \in H(div, \Omega) : q \cdot n = 0 \text{ on } \Gamma_N \} \\ H_{g,N}(div, \Omega) &:= \{ q \in H(div, \Omega) : q \cdot n = g \text{ on } \Gamma_N \} \end{aligned} \quad (9)$$

since they are crucial for the finite element approximation and for its further treatment. The unknowns for the mixed formualtion are approximated as $p \in H_{g,N}(div, \Omega)$ and $u \in L^2(\Omega)$. The weak formulation of (8) can be written as

$$\begin{aligned} &\int_\Omega p \cdot q\, dx + c^2 \rho \int_\Omega u\, div\, q\, dx = 0 \text{ for all } q \in H_{0,N}(div, \Omega) \\ &\int_\Omega v \cdot div\, p\, dx - \omega^2 \rho \int_\Omega v\, u\, dx = 0 \text{ for all } v \in L^2(\Omega). \end{aligned} \quad (10)$$

Nitsche's method for essential boundary conditions

Nitsche proposed a method for imposing essential boundary conditions weakly in the finite element method (FEM) approximation of elliptic problems defining and minimizing an energy functional, which penalizes simultaneously deviations from the solution of the PDE inside the domain and from the boundary conditions on the boundary. The Nitsche's method has been addressed many times as an improvement of the classical penalty method, overcoming the inconsistency with the original equation and the ill-condition system matrices yielded by the last. The method attracts attention because of the avoidance of additional unknowns or special interface meshes. Essentially, the enforcement of the boundary conditions is done via three additional boundary terms with respect to the initial equation. These terms can be categorized as terms responsible for providing consistency and symmetry, which contain in a weak form normal derivatives of the unknowns and the test functions, and stabilizing terms, which penalize the deviation from the boundary condition. The Helmholtz equations in terms of displacements and condition (4), which are imposed by the Nitsche's method, is expressed as

$$\begin{aligned} &\int_\Omega \nabla u \nabla q\, dx - \int_{\Gamma_D} q \frac{\partial u}{\partial n}\, ds - \int_{\Gamma_D} (u - u_0) \cdot n \frac{\partial q}{\partial n}\, ds + \\ &\qquad + \gamma \sum_{E \in \mathcal{G}} (u - u_0) \cdot nq - \frac{\omega^2}{c^2} \int_\Omega qu\, dx = 0, \end{aligned} \quad (11)$$

where $\gamma \sim \frac{2h_E}{|T|}$ is a mesh dependent penalty parameter.

Solution

Direct method

With respect to the handling of the Dirichlet boundary conditions in the context of the mixed Raviart-Thomas formulation, the investigated Nitsche's method is compared to an approach, referred as the "direct" method. The idea is to analyze the performance of the proposed Nitsche's approach and to evaluate its precision for this particular implementation, compared to the direct method. The so-called direct method makes use of the nature of the mixed formulation and introduces the boundary term containing the predefined values for u in the right hand side of the first equation from (10):

$$\int_\Omega p \cdot q\ dx + c^2 \rho \int_\Omega u\ div\ q\ dx = \int_{\Gamma_D} u_0 q \cdot n\ ds. \quad (12)$$

The resulting matrix form of the equation can be written as

$$\begin{pmatrix} B & C \\ C^T & K \end{pmatrix} \begin{pmatrix} x_\psi \\ x_u \end{pmatrix} = \begin{pmatrix} b_D \\ 0 \end{pmatrix}. \quad (13)$$

Nitsche's method

After applying the Nitsche's method to the weak mixed formulation, the terms involved are split, such that all terms including unknowns stay on the left hand side of the equation, while terms with known values on the boundaries are placed on the right hand side. The utilization of the method leads to a modification of the second equation from (10):

$$\int_\Omega v\, div\, p\, dx - \int_{\Gamma_D} v \frac{\partial u}{\partial n}\, ds - \int_{\Gamma_D} u \cdot n \frac{\partial v}{\partial n}\, ds + \gamma_0 \sum_{E\in\mathcal{G}} u \cdot nv$$
$$-\frac{\omega^2}{c^2}\int_\Omega vu\, dx = \int_{\Gamma_D} u_0 \cdot n \frac{\partial v}{\partial n}\, ds + \gamma_0 \sum_{E\in\mathcal{G}} u_0 \cdot nv, \quad (14)$$

the corresponding matrix equation is given as

$$\begin{pmatrix} B & C_N \\ C_N^T & K_N \end{pmatrix} \begin{pmatrix} x_\psi \\ x_u \end{pmatrix} = \begin{pmatrix} 0 \\ b_N \end{pmatrix}. \quad (15)$$

It should be noted that in both approaches for each element T the local stiffness matrix B is a diagonal matrix, but the matrix C is a vector. The motivation for the matrix structures originates from the spaces used for the approximation of the unknowns.

Numerical example over a unit square domain

For reason of comparisons the simple unit square domain in Fig. 2 is considered. The Dirichlet boundary condition at the right hand side of the domain is once imposed via the direct method and afterwards with the Nitsche's method. The results achieved in both cases are analyzed with respect to the deviation error from the prescribed boundary values. As can be seen from the matrix formulations (13) and (15) the vector of the unknowns consist of two components x_ψ and x_u. The x_ψ corresponds to the pressure calculation and the size of this vector is equal to the number of the mesh edges. The x_u components, which are related to the displacements, are constant over an element. In Fig. 3 and Fig. 4displacement and pressure results are shown, respectively,computed according to

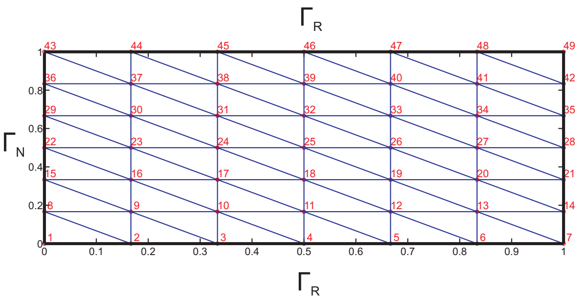

Figure 2: Unit square domain decomposition with boundary conditions.

$$u_h|_{Tl} = x_{ul} \text{ and } p_h = \sum_{k=1}^{N} x_{\psi_k}\psi_k, \quad (16)$$

where $l = 1, ..., L$ is the number of elements and $k = 1, ..., N$ is the number of edges in the discretization. Therefore, the physically meaningful (pressure) and the numerically practical (displacement) unknowns are obtained simultaneously.

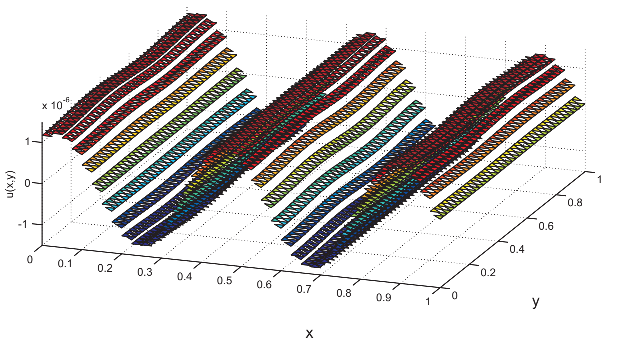

Figure 3: Element wise constant displacement with boundary condition $u_0 = 0$.

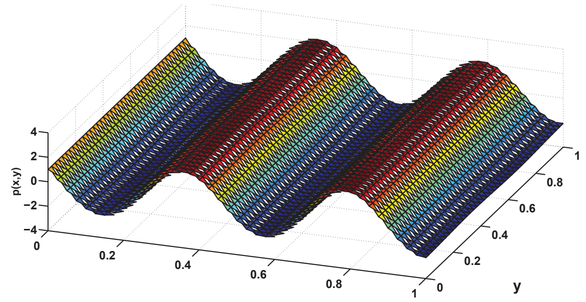

Figure 4: Pressure calculated as the additional unknown in the mixed formulation

Stability and error control

As mentioned before the stability issue in this approach originates from the choice of the penalty parameter γ. As reported by the authors in [3], the values for γ need to be taken close to the lower limit, due to stability consideration, since $\gamma \to \infty$ a locking problem is observed. A posteriori error control, based on the averaging technique for the Helmholtz problem is considered. The error estimation calculation on the boundary, where the Dirichlet boundary condition is prescribed involves $\frac{\partial u_0}{\partial s}\big|_E := \nabla u_0 \cdot \tau_E$, whit τ_E being a tangential unit vector to the exterior edge. In order to evaluate the influence of the penalty parameter on the error, a set of calculations are performed with increasing values for the penalty parameter. The initial value is chosen as $\gamma_0 = \frac{2c^2 h_E}{|T|}$ and larger values are the multiplication of γ_0 with the powers of ten. The errors calculated for four different penalty parameters are depicted in Fig. 5 along with the error obtained when the direct approach (denoted by the red line) is used for the enforcement of the essential boundary condition.

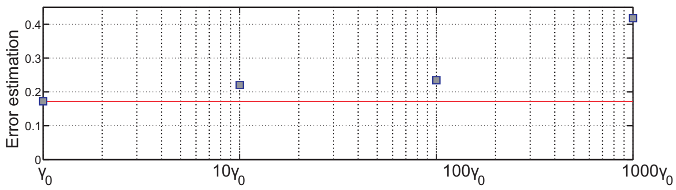

Figure 5: Error estimates for increasing penalty parameter

Conclusions and Outlook

The investigated Nitsche's approach for imposing essential boundary conditions for the mixed Raviart-Thomas finite element approximation of the Helmholtz equation in terms of displacement confirmed the claimed consistency and stability of the method. The application of the method for an academical example of 2D displacement based Helmholtz equation is proven to be comparable with the direct approach. The achieved results in a sense of an a posteriori error estimation tend to be penalty parameter dependent, which requires a careful handling of the parameter values. Considering the achieved results, it should be pointed out that dealing with nonconforming grids in this context could be an advantage, despite the computational efforts. The Raviart-Thomas finite elements have been proven to deliver spurious modesfree solution in terms of displacement, together with being well suited for the use of Nitsche's method for boundary conditions. They open interesting possibilities for the realisation of elasto-acoustic couplings, since for the Nitsche's method node-wise matching of the meshes on the interfaces between the domains is not required, a lot of freedom with respect to the choice of the discretization for the solid domains is given. As it has been suggested in [12] fluid-structure interaction could as well be modeled with the use of a combination between Crouzeix-Raviart and Raviart-Thomas finite elements based on the displacement-displacement formulation. In such a case the spurious modes free solution of the displacement based Helmholtz equation, achieved with the help of a Raviart-Thomas element domain discretization, could be coupled with enriched Crouzeix-Raviart elements used for the solid. Therefore the non-matching grids situation together with the above mentioned coupling is a quite promising area for further investigations.

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