Effect of structural resonances on LQR controlled mechanical systems

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Introduction

The design of a position controller on a mechanical system is an engineering challenge in three aspects. Firstly, most mechanical systems are non-linear and have multiple degrees of freedom (MDOF). In some cases, a mechanism can be designed such that it behaves linear in the controlled region and that the degrees of freedom are decoupled. In these cases, the position controller can be designed using the transfer function approach. The controller will be designed at hand of the open loop transfer function of the mechanical system. The designer pays attention to the loop gain at low frequencies and to the stability margins (phase and amplitude margins) which determine the controller bandwidth.

Secondly, most the mechanical systems are inertial systems. This implies that harmonic motion of the mechanism is in opposite phase resulting in a quasi unstable controlled system. As consequence, a differentiating action or a lead-lag action at the bandwidth frequency is necessary to stabilize the system.

Finally, structural resonances of the mechanical system are the most common cause for instability of the controlled system. The phase delay associated with such a resonance in combination with a control gain larger than 1 caused by the increase of the gain by the resonance amplitude will cause instability.

The controller design approach for multiple degrees of freedom mechanical systems starts from the state space description of the system. In state space, the system is represented as a set of first order differential equations. As mechanical systems normally are represented as second order systems, each degree of freedom will produce two states, which are displacement and velocity (when analysing the system with Newton or Lagrange method), or displacement and momentum (Hamilton method). If the set is non-linear, the set has to be linearised around a suitable work point before a linear controller can be designed. Then, the LQR-design procedure can be applied on the linearised set which results in a control matrix, which consists of a set of control gains between each state and each actuator [1]. The rank of the controller will be equal to the rank of the system. The LQR-design takes place at hand of a cost function which weights the control performance and the actuator effort. As the first order system is intrinsically stable, the criterion cannot take stability criteria into account. A new approach could be to develop a new criterion in which stability is included, based on a linearised MDOF model wherein the most dangerous resonances are included, where the control matrix only provides control gains on a subset of selected DOF's. The purpose of this paper is to demonstrate at hand of an example of a planar 2D-pick and place robot the necessity to develop such a new approach.

Modelling a planar 2D-pick and place robot.

Figure 1 presents a scheme of a 2-D robot which is typically used as pick-and-place robot for assembly tasks. The robot consists of an arm with a moment of inertia J_a around its rotation axis. On the arm, a carriage with a mass m and a moment of inertia J_c can slide. In this way, the tool connected to the carriage can reach every point within the circle determined by the arm length.

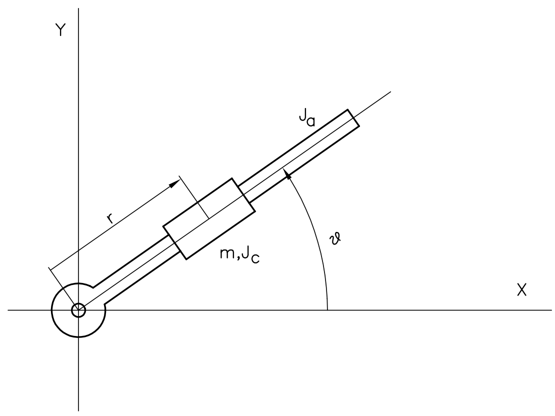

Figure 1: 2-D plane robot consisting of a carriage sliding over a rotating arm.

To obtain the set of first order equations, Hamilton's method is used [2]. This method has several advantages. 1) Hamilton's method results directly in a set of first order equations, 2) The equations of motion results from an analysis of the motion of the system is in stead of the analysis of the forces, 3) All the forces and moments, including Coriolis forces and gyroscopic moments, appear automatically from the method.

The procedure to obtain Hamilton's equations of motion described below can be applied for any mechanism.

1. First, the number of DOF's have to be determined, in this case there are two: the arm angle θ and carriage position r.

2. The kinetic energy E_k, potential energy E_p and dissipated power P are determined from its motion,

resulting in:

$$
\begin{aligned}
E_k &= \tfrac{1}{2}\, J_a\, \dot{\theta}^2 + \tfrac{1}{2}\, m\, \dot{r}^2 + \tfrac{1}{2}\, m\, (r\, \dot{\theta})^2 + \tfrac{1}{2}\, J_c\, \dot{\theta}^2 \\
&= \tfrac{1}{2}\, (J_a + J_c + m\, r^2)\, \dot{\theta}^2 + \tfrac{1}{2}\, m\, \dot{r}^2 \\
&= \tfrac{1}{2}\, (J + m\, r^2)\, \dot{\theta}^2 + \tfrac{1}{2}\, m\, \dot{r}^2 \quad \text{with} \quad J = J_a + J_c \\
E_p &= 0 \\
P &= 0
\end{aligned} \tag{1}
$$

3. The momentums p_r and p_θ in both degrees of freedom θ and r are determined from the kinetic energy E_k:

$$
\begin{aligned}
p_\theta &= \frac{\partial E_k}{\partial \dot{\theta}} = (J + m\, r^2)\, \dot{\theta} \\
p_r &= \frac{\partial E_k}{\partial \dot{r}} = m\, \dot{r}
\end{aligned} \tag{2}
$$

The Hamiltonian H has to be expressed in displacement and momentum, so the velocities have to be eliminated from the momentums:

$$
\begin{aligned}
\dot{\theta} &= \frac{p_\theta}{J + m\, r^2} \\
\dot{r} &= \frac{p_r}{m}
\end{aligned} \tag{3}
$$

4. The Hamiltonian H is the sum of the kinetic and the potential energy, expressed in displacement and momentum,

$$
\begin{aligned}
H &= E_k + E_p \\
&= \frac{1}{2}\, \frac{p_\theta^2}{J + m\, r^2} + \frac{1}{2}\, \frac{p_r^2}{m}
\end{aligned} \tag{4}
$$

5. For each degree of freedom q, the set of equations of motion will be determined:

$$
\begin{cases}
\dot{q} = \frac{\partial H}{\partial p} \\
\dot{p} = F_q - \frac{\partial H}{\partial q}
\end{cases} \tag{5}
$$

resulting for de degrees of freedom θ and r in:

$$
\begin{cases}
\dot{r} = \dfrac{p_r}{m} \\
\dot{\theta} = \dfrac{p_\theta}{J + m\, r^2} \\
\dot{p_r} = F_r + p_\theta^2\, \dfrac{m\, r}{(J + m\, r^2)^2} \\
\dot{p_\theta} = F_\theta
\end{cases} \tag{6}
$$

This result is the set of first order differential equations of motion for the 2D-robot.

In order to design a controller for the robot, the next step is to linearise the set equations of motion in a suitable working point, i.e. the determination of the system matrix $\mathbf{A}$, which is the Jacobian:

$$
\mathbf{A} = \begin{bmatrix}
\frac{\partial \dot{r}}{\partial r} & \frac{\partial \dot{r}}{\partial \theta} & \frac{\partial \dot{r}}{\partial p_r} & \frac{\partial \dot{r}}{\partial p_\theta} \\
\frac{\partial \dot{\theta}}{\partial r} & \frac{\partial \dot{\theta}}{\partial \theta} & \frac{\partial \dot{\theta}}{\partial p_r} & \frac{\partial \dot{\theta}}{\partial p_\theta} \\
\frac{\partial \dot{p_r}}{\partial r} & \frac{\partial \dot{p_r}}{\partial \theta} & \frac{\partial \dot{p_r}}{\partial p_r} & \frac{\partial \dot{p_r}}{\partial p_\theta} \\
\frac{\partial \dot{p_\theta}}{\partial r} & \frac{\partial \dot{p_\theta}}{\partial \theta} & \frac{\partial \dot{p_\theta}}{\partial p_r} & \frac{\partial \dot{p_\theta}}{\partial p_\theta}
\end{bmatrix} \tag{7}
$$

After calculation of all these derivatives in the work point r_0, p_{r0}, θ_0 and $p_{\theta 0}$, the linear state equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ will be:

$$
\begin{bmatrix} \dot{r} \\ \dot{\theta} \\ \dot{p_r} \\ \dot{p_\theta} \end{bmatrix}
=
\begin{bmatrix}
0 & 0 & \frac{1}{m} & 0 \\
-\frac{2\, m\, r_0\, p_{\theta 0}}{(J + m\, r_0^2)^2} & 0 & 0 & \frac{1}{J + m\, r_0^2} \\
\frac{m\, p_{\theta 0}\, (J - 3\, m\, r_0^2)}{(J + m\, r_0^2)^3} & 0 & 0 & \frac{2\, m\, r_0\, p_{\theta 0}}{(J + m\, r_0^2)^2} \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} r \\ \theta \\ p_r \\ p_\theta \end{bmatrix}
+
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} 0 \\ 0 \\ F_r \\ F_\theta \end{bmatrix}
$$

Once the set of Hamiltonian equations and the linearised set in the form $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u}$ are obtained, the controller design can be carried out.

The LQR design method is based on minimizing a cost criterion:

$$
J = \int_0^\infty (\mathbf{x}^T\, \mathbf{Q}\, \mathbf{x} + \mathbf{u}^T\, \mathbf{R}\, \mathbf{u})\, dt \tag{8}
$$

The performance is weighted by the square matrix $\mathbf{Q}$. By pre-multiplication by the state vector $\mathbf{x}^T$ en post-multiplication by $\mathbf{x}$, the result is one scalar quantity. In the same way, the control effort is weighted by the square matrix $\mathbf{R}$ and generates also a scalar. By integration over time, the criterion balances performance and control effort over the total period of time the controller is active.

This procedure is implemented in many numerical programs, such as MatLab or Octave. After a few iterations for $\mathbf{Q}$ and $\mathbf{R}$, a satisfactory controller results. The obtained controller will then be simulated in closed loop on the non-linear system. Figure 2 (a) presents the carriage

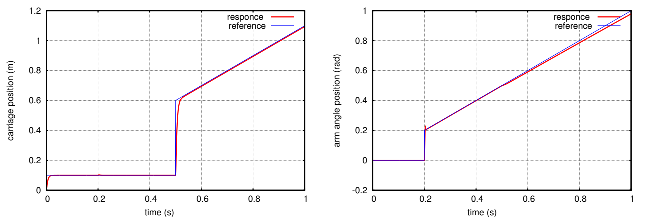

Figure 2: Desired path (thin blue line) and responce (thich red line) of the idealized 2D plane robot **(a)** r-position of the carriage on the arm. **(b)** angle position of the arm.

position on the arm and (b) presents the arm angle. In blue thin line, the desired path is plotted and in thick red line, the resulting path of the carriage motion is plotted. At 0.5 s, when the carriage starts to move outwards, the deviation between the desired angle and the resulting angle grows due to the continuously increase of the robot inertia by the carriage outward motion. The resulting control performance is satisfactory for this system. Actually, the control performance can be increased further without becoming unstable, the sampling period is the limit which causes instability ultimately. The idealized mechanical system in intrinsically stable.

Modelling a planar 2D-pick and place robot with a disturbing resonance.

A disturbing resonance is a resonance of the system which is not included in the model which is used to design the controller. In mechanical systems, such resonances are unavoidable due to the distributed mass and elasticity in the components. These resonances form the real limit on the machine performance. In general, expanding the controller with additional degrees of freedom does not generate additional performance and the problems have to be solved in the mechanical construction. It is an illusion that an ill designed mechanical system can be compensated by a more advanced controller.

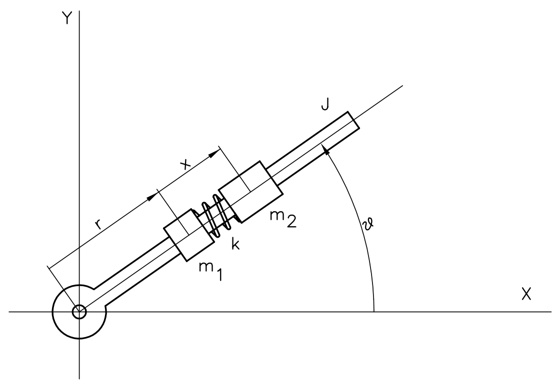

Figure 3: 2-D plane robot with carriage resonance.

Identifying resonances in the mechanical structure is one of the keys to design mechanical systems suitable for high performance control. A controller can remain stable when only one disturbing resonance is in the control bandwidth, but two resonances in the control bandwidth is guaranteed unstable. In this section, a disturbing resonance is included in the non-linear model and its effect is demonstrated at hand of the 2-D robot example. To be safe, the two resonances with the lowest frequency should be included in the non-linear model for each degree of freedom to be controlled. By carrying out this analysis, the mechanical construction of the system can be investigated before prototyping and proper choices for mechanical components such as driving components, connection parts, etc... can be made.

Figure 3 represent the 2-D robot with a disturbing resonance in the carriage. Typically, such resonance can be caused by a belt drive or an elastic coupling. As consequence, the carriage mass is divided in two parts with mass m_1 and m_2, connected by the spring k. The spring connection is supposed to have a viscous damping c. When the spring stiffness $k \to \infty$, the original situation presented in figure 1 is obtained. In this analysis, the measurement of the carriage position $r+x$ takes place on the carriage with mass m_2 and the actuator force F_r acts on the actuator mass m_1, so the resonance occurs between the position sensor and the actuator.

The equations of motion are again determined using the Hamilton method:

1. There are now three degrees of freedom: the robot arm angle θ, the carriage position r and the spring deformation x.

2. So, the kinetic energy E_k, potential energy E_p and dissipated power P will be:

$$\begin{aligned} E_k &= \tfrac{1}{2}\, J\, \dot{\theta}^2 + \tfrac{1}{2}\, m_1\, \dot{r}^2 + \tfrac{1}{2}\, m_2\, (\dot{r}+\dot{x})^2 \\ &\quad + \tfrac{1}{2}\, m_1\, (r\,\dot{\theta})^2 + \tfrac{1}{2}\, m_2\, ((r+x)\,\dot{\theta})^2 \\ E_p &= \tfrac{1}{2}\, k\, x^2 \\ P &= \tfrac{1}{2}\, c\, \dot{x}^2 \end{aligned} \tag{9}$$

3. The momentums p_r, p_θ and p_x in all degrees of freedom r, θ and x are determined from the kinetic energy E_k :

$$\begin{aligned} p_\theta &= \frac{\partial E_k}{\partial \theta} = (J + m_1\, r^2 + m_2\, (r+x)^2)\, \dot{\theta} \\ p_r &= \frac{\partial E_k}{\partial r} = m_1\, \dot{r} + m_2\, (\dot{r}+\dot{x}) \\ p_x &= \frac{\partial E_k}{\partial x} = m_2\, (\dot{r}+\dot{x}) \end{aligned} \tag{10}$$

The Hamiltonian H has to be expressed in displacement and momentum, so the velocities have to be eliminated from the momentums:

$$\begin{aligned} \dot{\theta} &= \frac{p_\theta}{J + m_1\, r^2 + m_2\, (r+x)^2} \\ \dot{r} &= \frac{p_r - p_x}{m_1} \\ \dot{x} &= \frac{p_x}{m_2} - \frac{p_r - p_x}{m_1} \end{aligned} \tag{11}$$

4. The Hamiltonian H is the sum of the kinetic and the potential energy, expressed in displacement and momentum:

$$\begin{aligned} H &= E_k + E_p \\ &= \frac{1}{2}\, \frac{p_\theta^2}{J + m_1\, r^2 + m_2\, (r+x)^2} + \frac{1}{2}\, \frac{(p_r - p_x)^2}{m_1} \\ &\quad + \frac{1}{2}\, \frac{p_x^2}{m_2} + \frac{1}{2}\, k\, x^2 \end{aligned} \tag{12}$$

5. For each degree of freedom q, the set of equations of motion will be determined:

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = F_q - \frac{\partial H}{\partial q} - \frac{\partial P}{\partial \dot{q}} \end{cases} \tag{13}$$

resulting for de degree of freedoms r, θ and x:

$$\begin{cases} \dot{r} = \dfrac{p_r - p_x}{m_1} \\ \dot{\theta} = \dfrac{p_\theta}{J + m_1\, r^2 + m_2\,(r+x)^2} \\ \dot{x} = \dfrac{p_x}{m_2} - \dfrac{p_r - p_x}{m_1} \\ \dot{p_r} = F_r + p_\theta^2\, \dfrac{m_1\, r + m_2\,(r+x)}{(J + m_1\, r^2 + m_2\,(r+x)^2)^2} \\ \dot{p_\theta} = F_\theta \\ \dot{p_x} = p_\theta^2\, \dfrac{m_2\,(r+x)}{(J + m_1\, r^2 + m_2\,(r+x)^2)^2} - k\,x - c\,\dot{x} \end{cases}$$

This set of equations is used for the non-linear simulation of the controlled system with the disturbing resonance. The applied controller is the same as the one obtained for the ideal system from the previous section. Only the set of non-linear equations has been replaced by the expanded set containing the disturbing resonance.

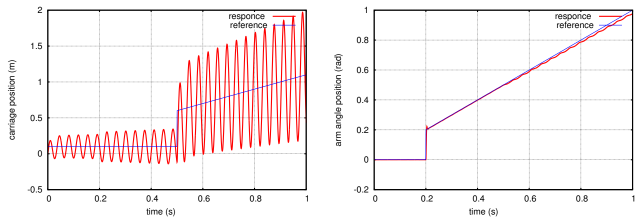

Figure 4: Desired path (thin blue line) and responce (thich red line) of the 2D plane robot with carriage resonance **(a)** r-position of the carriage on the arm. **(b)** angle position of the arm.

Figures 4 display the motion of the carriage and the arm of the disturbed system when the control loop is closed. The performance is clearly affected in a negative way due to the disturbing resonance. The carriage motion has become unstable due to the disturbing resonance. The arm motion remains stable, however the motion of the carriage has an effect on the arm motion. If the mechanical system cannot be redesigned, the control performance has to be decreased to stabilize the system again. The re-

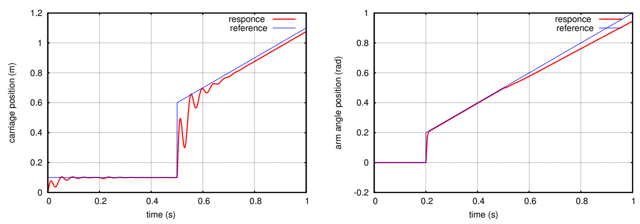

Figure 5: Desired path (thin blue line) and responce (thich red line) of the 2D plane robot with carriage resonance **(a)** r-position of the carriage on the arm. **(b)** angle position of the arm.

sult of decreasing the performance weighting matrix Q is presented in figure 5. The stabilized system still exhibits some oscillation in the carriage response. The position error and the convergence speed of the controlled system is now larger than in the idealized system presented in figure 2.

Proposal for a new LQR criterion for mechanical systems

From the previous sections, it appears that the LQR-cost criterion is not satisfactory for mechanical system which exhibit disturbing resonances. However, the LQR-criterion works very well for undisturbed systems with multiple degrees of freedom. After a few iterations, a suitable controller can be designed. It would be desirable to have such a criterion at our disposal for systems with disturbing resonances. A way to set up such a criterion could be as follows:

1. The equations of motion including the disturbing resonances have to be set up. This can be done using the Newton, Lagrange or Hamilton equations. Additional to the number of degrees of freedom of the desired motion of the mechanical system, each resonance adds an additional degree of freedom.
2. The controller has to be designed on a subset of the equations of motion of the mechanical system, in which only the degrees of freedom of the desired motion are incorporated. This subset will then be linearised and represented as the linear state equation $\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u}$.
3. A new control criterion has to be developed, which is based on stability in stead of cost. The criterion should take into account the full set of equations of motion of the mechanical system. It would be desirable that lead-lag actions could be incorporated in the controller. The development of such a control criterion is the aim for further research.
4. The resulting controller will then be simulated on the full set of equations of motion of the mechanical system to check its performance.

Conclusion.

In this paper it is demonstrated at hand of a 2D-pick and place robot that the cost criterion which is used in the LQR control design method which takes only control performance and actuator effort into account is not sufficient to have optimal controllers when control stability is important. Therefore, particularly for mechanical multiple degrees of freedom systems, a control design criterion which takes stability into account should be developed. The control design should be carried out on a subset of the equations of motion which contains the degrees of freedom to be controlled. The stability has to be evaluated on the full set of equations of motion, which contains also the degrees of freedom associated to the disturbing resonances. The result would be a design method which can be applied as efficient as the LQR controller design, wherein stability is evaluated in stead of a control cost.

References

[1] D.H. Owens: Multivariable and Optimal Systems, Academic press, London, 1981.

[2] D.A. Wells: Theory and problems of Lagrangian dynamics, Schaum Outline Series, New York, 1967.