Explicit Sound Field Synthesis Driving Functions in the Spatial Domain

Gergely Firtha, Peter Fiala
Budapest University of Technologies and Economics, 1117 Budapest, Hungary, Email: firtha,fiala@hit.bme.hu

Introduction

Sound Field Synthesis (SFS) aims at the physical reconstruction of an arbitrary target sound field over an extended listening area by employing a densely spaced loudspeaker distribution termed, the *secondary source distribution (SSD)*. The SSD is fed with a properly derived *driving function* so that the resultant field of the individual SSD elements coincides with the target sound field in the intended receiving area [1].

Two main methodologies exist for the calculation of the SSD driving functions: *Impicit solutions*—e.g. Wave Field Synthesis—extract the required driving functions from a suitable boundary integral formulation of the target sound field, containing the required driving functions implicitly [2, 3, 4].

Explicit solution obtains the direct solution of the inverse problem. Once a spectral decomposition of the SSD elements' wave field and the target field is known in a chosen geometry, the driving function spectrum may be obtained by comparing the corresponding spectral coefficients of the involved sound fields taken on a control surface [5]. The explicit solution for linear and planar SSDs, employing spatial Fourier transforms for spectral decomposition is often referred to as *Spectral Division Method (SDM)* [6, 7, 8] Since the explicit solution formulates the driving function in the spectral/wavenumber domain, the spatial solution is obtained in the form of an inverse spectral integral. The direct implementation of this integral formulation in practical SFS scenarios is not feasible. Still, it serves as a reference solution with no approximations involved.

This contribution presents a high-frequency approximation of the explicit SFS driving functions, starting out from the explicit solution in a linear SSD geometry. The derivation utilizes the *Stationary Phase Approximation (SPA)* resulting in driving functions, requiring only the target sound pressure measured on an arbitrarily chosen reference curve. Over this reference curve amplitude correct synthesis is optimized. Besides presenting a novel SFS formulation, suitable for direct implementation, the presented driving functions also allow the direct comparison of the implicit and explicit solutions.

Theoretical basics

Explicit driving functions for a linear SSD

Consider a linear SSD at $\mathbf{x}_0 = [x_0, 0, 0]^{\mathrm{T}}$, consisting of a continuous distribution of 3D point sources, weighted by the driving function $D(x_0, \omega)$. The synthesized field is given by the sum of the radiated field of individual SSD elements, in the form of a spatial convolution

$$P(\mathbf{x}, \omega) = \int_{-\infty}^{\infty} D(x_0, \omega) G(x - x_0, y, z, \omega) \mathrm{d}x_0, \qquad (1)$$

with the kernel of convolution $G(\mathbf{x}, \omega)$ representing the transfer function of the individual SSD elements, generally being the 3D free-field Green's function describing the field of an acoustic point source. Performing a Fourier-transform along x_0 transforms the convolution into a spectral multiplication, yielding the synthesized field in the wavenumber domain:

$$\tilde{P}(k_x, y, 0, \omega) = \tilde{D}(k_x, \omega) \tilde{G}(k_x, y, 0, \omega), \qquad (2)$$

with restricting the investigation to the $z = 0$ *synthesis plane.*

Hence, the wavenumber content of the driving functions are obtained as the ratio of the target sound field spectrum and the Green's function spectrum measured along a fixed y-coordinate, and the spatial solution is obtained in the form of a corresponding spatial inverse Fourier transform

$$D(x_0, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\tilde{P}(k_x, y, 0, \omega)}{\tilde{G}(k_x, y, 0, \omega)} \mathrm{e}^{-\mathrm{j} k_x x_0} \mathrm{d}k_x. \qquad (3)$$

In order to give an approximation for the general SDM driving function (3) the stationary phase approximation (SPA) is applied.

The Stationary Phase Approximation

As a basic tool of asymptotic analysis the method yields an approximate solution for integrals of the form

$$I = \int_{-\infty}^{\infty} F(x)\, \mathrm{e}^{\mathrm{j}\phi(x)}\, \mathrm{d}x, \qquad (4)$$

when $\mathrm{e}^{\mathrm{j}\phi(x)}$ is highly oscillating and $F(x)$ is comparably slowly varying.

A rigorous derivation of the SPA based on integration by parts is given in [9, 10]. More informally the method relies on the second order truncated Taylor series of the exponent around x^*, where $\phi'_x(x^*) = 0$ and $\phi''_{xx}(x^*) \neq 0$, with $\phi'_x(x)$ denoting the derivative with respect to x. The critical point x^* is termed the *stationary point.* The SPA assumes that where the phase varies, i.e. $\phi'_x(x) \neq 0$, the integral of rapid oscillation cancels out, and the greatest contribution to the total integral comes from the immediate surroundings of the stationary point. Moreover in the proximity of the stationary point $F(x)$ can be regarded as constant with the value $F(x^*)$.

With these considerations—supposing also only one stationary point in the integration path—the integral can be approximated as

$$I \approx \sqrt{\frac{2\pi}{|\phi''_{xx}(x^*)|}} \mathrm{e}^{\mathrm{j}\frac{\pi}{4}\,\mathrm{sgn}\left(\phi''_{xx}(x^*)\right)} F(x^*)\,\mathrm{e}^{+\mathrm{j}\phi(x^*)}, \tag{5}$$

hence by the function value around the stationary point $F(x^*)\,\mathrm{e}^{\mathrm{j}\phi(x^*)}$ itself, along with a curvature correction factor.

The local wavenumber vector

When the SPA is applied to Fourier integrals an expressive physical interpretation can be given, by introducing the local wavenumber vector concept. Consider an arbitrary steady state harmonic sound field, written in a general polar form

$$P(\mathbf{x},\omega) = A_P(\mathbf{x},\omega)\mathrm{e}^{\mathrm{j}\phi_P(\mathbf{x},\omega)}, \tag{6}$$

with $A_P(\mathbf{x},\omega), \phi_P(\mathbf{x},\omega) \in \mathbb{R}$ being its amplitude and phase functions. The dynamics of the wave propagation is described by the phase function of the sound field. Here we introduce the *local wavenumber vector*, defined by the gradient of the phase function

$$\mathbf{k}^l_P(\mathbf{x}) = -\nabla\phi_P(\mathbf{x},\omega). \tag{7}$$

The local wavenumber vector points in the direction of the maximum phase advance, i.e. it is perpendicular to the wave front in an arbitrary position pointing in the local wave propagation direction. This is obviously a local plane wave approximation of arbitrary sound fields —stemming from the first order Taylor's approximation of (6)—, for which the *local dispersion relation* holds:

$$\left(\frac{\omega}{c}\right)^2 = |\mathbf{k}^l_P(\mathbf{x})|^2 = k^l_{x,P}(\mathbf{x})^2 + k^l_{y,P}(\mathbf{x})^2 + k^l_{z,P}(\mathbf{x})^2. \tag{8}$$

The length of the vector is frequency dependent, which remains unindicated in the followings for the sake of brevity.

For the sake of simplicity in the present treatise we restrict the investigation to strictly non-converging (i.e. to diverging/outgoing) wave fronts, for which we require that the second partial derivatives of the phase function are nonpositive. This holds trivially for simple sound fields, e.g. fields of point/line sources and plane waves. The possibility of synthesizing focused sources is therefore excluded from the present investigation. For an illustration of the introduced vector quantity see Figure 1.

Explicit driving functions in the spatial domain

This section presents the asymptotical approximation of the general SDM driving functions (3) by applying the SPA. The derivation starts with the approximation of the target field and Green's function spectra in the general SDM driving functions given by (3). For the sake of

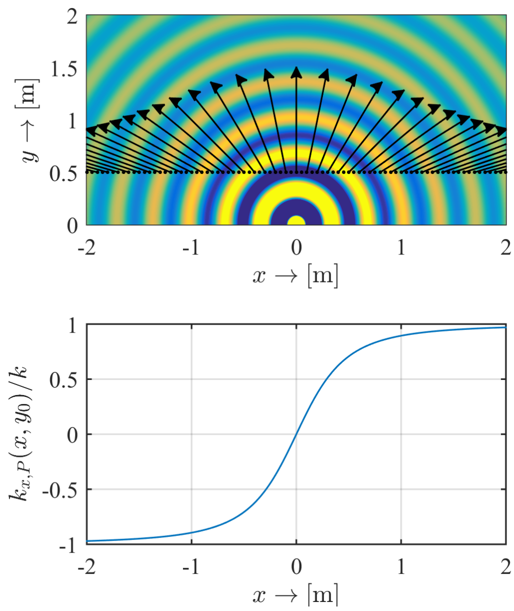

Figure 1: Illustration of the wavenumber vector, in case of a 3D point source, taken along $y_0 = 0.5$ m. The wavenumber vector, and the x component of the local wavenumber are normalized by $k = \frac{\omega}{c}$.

brevity ω and z dependencies are suppressed. The latter is defined at $z = 0$ throughout the followings. By definition the wavenumber content of the target sound field is obtained via a forward Fourier transform with $A_P, \phi_P \in \mathbb{R}$

$$P(k_x, y) = \int_{-\infty}^{\infty} A_P(x,y)\,\mathrm{e}^{\mathrm{j}\phi_P(x,y)}\,\mathrm{e}^{\mathrm{j}k_x x}\mathrm{d}x, \tag{9}$$

and for the Green's function in the same manner.

Under high-frequency assumptions the integrals may be approximated by evaluation around its stationary point $x^*_P(k_x)$, where

$$-\phi'_x(x^*(k_x), y) = k^l_x(x^*(k_x), y) = k_x \tag{10}$$

holds. Since the forward Fourier transform can be interpreted as projection onto spectral plane waves, that's propagation direction is determined by k_x (as long as $k_z = 0$) this means, that *the stationary points—i.e. the greatest contribution— is found for the Fourier integral for an arbitrary k_x component, where the local propagation direction of the field to transform coincides with that of the corresponding spectral plane wave.*

Supposing, that $x^*_P(k_x)$ and $x^*_G(k_x)$ are the stationary positions for the corresponding integrals, i.e.

$$k^l_{x,P}(x^*_P(k_x), y) = k^l_{x,G}(x^*_G(k_x), y) = k_x \tag{11}$$

holds, and accounting for the negative second derivatives—since both P and G are non-converging waves—their spectra can be approximated by the SPA, and the asymptotic approximation of the SDM driving

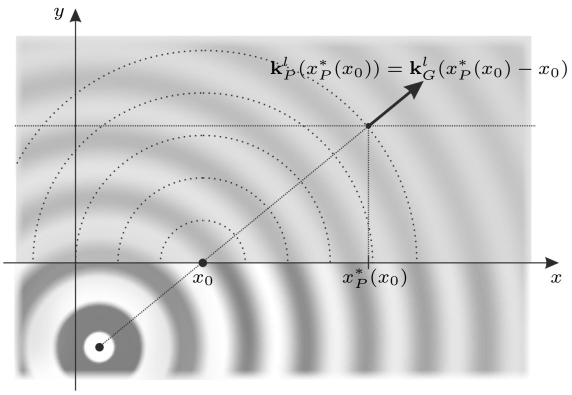

Figure 2: Illustration of the evaluation position $x^*_P(x_0)$ (and $x^*_G(x_0)$) as the function of x_0. For a given SSD position x_0 the stationary positions at y, where the virtual field propagation direction coincides with that of the Green's function translated into x_0.

functions on a given spectral component reads

$$D(k_x, y) \approx \sqrt{\frac{|\phi''_{G,xx}(x^*_G(k_x), y)|}{|\phi''_{P,xx}(x^*_P(k_x), y)|}} \frac{P(x^*_P(k_x), y)}{G(x^*_G(k_x), y)} \mathrm{e}^{\mathrm{j}k_x \cdot (x^*_P(k_x) - x^*_G(k_x))}. \tag{12}$$

The second step of derivation approximates the inverse Fourier integral of the asymptotical SDM driving functions (12) by a further application of the SPA. The stationary point for a given position x_0 relates the evaluation points x^*_P and x^*_G directly to the actual SSD coordinate x_0, therefore the intermediate stationary wavenumber (k^*_x) dependency may be omitted. Along with the definition of the forward transform stationary points (11), the stationary positions satisfy

$$k^l_{x,P}(x^*_P(x_0), y) = k^l_{x,G}(x^*_G(x_0), y), \tag{13}$$

where

$$x^*_P(x_0) - x_0 = x^*_G(x_0) \tag{14}$$

holds. This result states, that *for a given SSD coordinate x_0 the evaluation point x^*_P is found, where the local propagation direction of the target field P coincides with that of a point source positioned at $[x_0,\ 0,\ 0]^{\mathrm{T}}$ measured on the reference line.* For an illustration refer to Figure 2.

For the sake of brevity the evaluation point is denoted by $x^*_P \rightarrow x^*$. Around the stationary point the inverse transform of (12) may be evaluated. The required second derivatives may be expressed by implicit differentiation and as a result one obtains the spatial asymptotic SDM driving functions

$$D(x_0) \approx \sqrt{\frac{\left|\phi''_{G,xx}(x^*(x_0) - x_0, y)\right|^2}{\left|\phi''_{P,xx}(x^*(x_0), y) - \phi''_{G,xx}(x^*(x_0) - x_0, y)\right|}} \sqrt{\frac{\mathrm{j}}{2\pi}} \frac{P(x^*(x_0), y)}{G(x^*(x_0) - x_0, y)}, \tag{15}$$

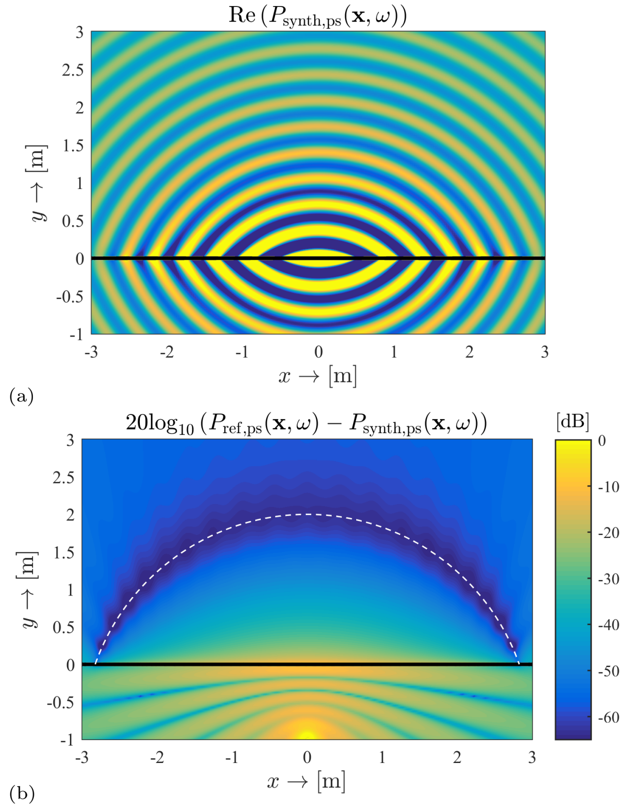

Figure 3: Synthesis of a 3D point source located at $\mathbf{x}_{\mathrm{s}} = [0,\ -1,\ 0]^{\mathrm{T}}$, oscillating at $\omega_0 = 2\pi \cdot 1$ krad/s. The synthesis is referenced on a circle around the virtual source, with a radius of $R_{\mathrm{ref}} = 3$ m denoted by white dashed line.

where $k^l_{x,P}(x^*(x_0)) = k^l_{x,G}(x^*(x_0) - x_0)$ holds. This result states, that an arbitrary sound field may be synthesized by finding the positions along the reference line, where the propagation direction/wavefront of the target field matches the field of the actual SSD elements. In this stationary position the driving functions are obtained by the ratio of the target field and the actual SSD element, corrected by the factor, containing the wavefront curvature at the same position. Therefore the explicit, global solution can be approximated by simple local wavefront matching.

One important fact is pointed out here: although having derived the above driving functions in terms a forward an inverse spatial Fourier transform along a straight line, there is no restriction on the y-coordinate of the stationary point in (15) due to the local approximations involved. This means, that an arbitrary referencing curve may be defined as $\mathbf{x}^*(x_0)$, and the driving functions can be calculated by finding the stationary positions $k^l_{x,P}(\mathbf{x}^*(x_0)) = k^l_{x,G}(\mathbf{x}^*(x_0) - \mathbf{x}_0)$ along this curve. Evaluating the driving functions in the stationary positions will result in amplitude correct synthesis along the reference curve within the validity of the SPA.

In the following a simple example is presented in order to demonstrate the validity of the spatial SDM driving functions.

Application Example

Consider the synthesis of a 3D point source, referencing the synthesis to a circle around the virtual point source. For the sake of simplicity the source is located at $\mathbf{x}_s = [0,\ y_s,\ 0]^T$. Along with the equation describing the reference curve $\mathbf{x}^*(x_0) = [x^*(x_0),\ y^*(x_0),\ 0]^T$ the stationary points satisfy the following equations

$$\frac{x^*(x_0)}{\sqrt{x^*(x_0)^2 + (y^*(x_0) - y_s)^2}} = \frac{x^*(x_0) - x_0}{\sqrt{(x^*(x_0) - x_0)^2 + y^*(x_0)^2}}, \tag{16}$$

$$x^*(x_0)^2 + (y^*(x_0) - y_s)^2 = R_{\text{ref}}^2. \tag{17}$$

The solution of the equations is given by

$$x^*(x_0) = x_0 \frac{R_{\text{ref}}}{\sqrt{x_0^2 + y_s^2}} \tag{18}$$

$$y^*(x_0) = y_s \left(1 - \frac{R_{\text{ref}}}{\sqrt{x_0^2 + y_s^2}}\right). \tag{19}$$

Substituting into (15) yields the explicit driving function in the spatial domain referencing the synthesis on a circle.

Investigating Figure 3 verifies, that the synthesis is optimized on the prescribed reference curve exhibiting a minimum in the amplitude error distribution.

Conclusion

This contribution presented an asymptotical approximation of the well-known explicit SFS solution, the Spectral Division Method. The derivation utilized the stationary phase approximation to approximate both forward and inverse Fourier transforms, resulting in loudspeaker driving functions purely in the spatial domain. The novel driving functions require the target sound pressure measured on an arbitrary receiver curve at which an amplitude correct synthesis is ensured within the validity of the SPA. The key concept of the driving function evaluation is finding the stationary position on the prescribed reference curve at which the local propagation direction of the target sound field coincides with that of the actual SSD element. The presented solution therefore directly accomplishes wave front matching.

One important conclusion should be drawn: the target sound field on the reference curve may be formulated in the form of a 2.5D Rayleigh integral, written in terms of the target field gradient measured along the SSD. Applying this formulation the presented solution results in the unified 2.5D WFS driving functions, presented by the authors recently in [11]. This therefore proves, that the general implicit solution is the high-frequency approximation of the general explicit solution for an arbitrary target sound field, which fact had been proven for specific target sound fields in the related literature [12, 13, 14]. This finding also verifies, that the presented driving functions are adequate for an arbitrary curved convex SSD within the validity of the Kirchhoff approximation.

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