Numerical Integration for the Isogeometric Boundary Element Method

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Introduction

The Boundary Element Method (BEM) is able to solve acoustical problems in the frequency domain, whereby only the surface is required to compute the desired values in the complete domain. The common procedure for numerical simulations starts with description of the geometry in CAD software. Usually, these geometries are approximated by Lagrange elements that leads to a discretization error. Afterwards, the numerical solution is able to compute the sound pressure. The Isogeometric Analysis (IGA) [1] eliminates the geometry approximation and uses the CAD description directly in the numerical method. Hence, the discretization error is removed and other steps have a larger influence, as for instance the numerical integration. The presented contribution deals with the importance of the numerical integration for the Isogeometric Boundary Element Method. Firstly, the formulation is explained and, secondly, the described routines are investigated for some representative examples.

Boundary Element Method

In the frequency domain, the Helmholtz equation

$$\nabla^2 p = -k^2 p \tag{1}$$

is the governing partial differential equation that describes the propagation of waves. The equation consists of the sound pressure p and the wavenumber $k = \frac{\omega}{c}$ with the speed of sound c. A transformation onto the surface Γ leads to the conventional boundary integral equation (CBIE)

$$c(\mathbf{x})p(\mathbf{x}) = \int_\Gamma \left[G(\mathbf{x},\mathbf{y})q(\mathbf{y}) - \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial \mathbf{n_y}} p(\mathbf{y}) \right] \mathrm{d}\Gamma_y \ , \tag{2}$$

where $\mathbf{x}$ is the receiver point, $\mathbf{y}$ is the source point, $\mathbf{n_y}$ is the normal vector at $\mathbf{y}$, and $G(\mathbf{x},\mathbf{y})$ is the fundamental solution. For the three dimensional case this fundamental solution reads

$$G(\mathbf{x},\mathbf{y}) = \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|} \ . \tag{3}$$

The boundary values $p(\mathbf{y})$ and $q(\mathbf{y})$ are discretized by a supporting point f_m with its assigned ansatzfunction $\phi_m(\xi)$

$$f(\xi) = \sum_{m=1}^{M} \phi_m(\xi) f_m \ . \tag{4}$$

The definition of the ansatzfunction leads to the used concept, either the classical Lagrange concept or the new IGA concept. A drawback of the CBIE is its non-uniqueness problem that corresponds to the occurrence of the so called spurious eigenfrequencies. The problem can be healed by the Burton-Miller formulation [5], which is based on the linear combination of the CBIE and the hypersingular boundary integral equation (HBIE)

$$c(\mathbf{x})\, q(\mathbf{x}) = \int_\Gamma \left[\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial \mathbf{n_x}} q(\mathbf{y}) - \frac{\partial^2 G(\mathbf{x},\mathbf{y})}{\partial \mathbf{n_x} \partial \mathbf{n_y}} p(\mathbf{y}) \right] \mathrm{d}\Gamma_y \ . \tag{5}$$

The difficulty of this equation is the strongly and hypersingular term, which require special integration routines. An interesting procedure to overcome this problem for the Isogeometric BEM is explained in a following section. The system of equation is created by a collocation method. An overview with a focus on iterative solvers for BEM systems is given in [2].

Isogeometric Analysis

In an Isogeometric Analysis the description of the geometry shall be used as the representation of the field variables. In the case of a BEM, these are the boundary values. The common standard in the CAD industry are NURBS that allow a rather simple description of complex shapes, as spheres or other roundings. NURBS surfaces are defined as patches that can consist of different elements. In general, the ansatzfunctions are defined on these patches and can differ between the elements. Additionally, the functions are defined recursively, which increases the numerical complexity for the evaluation significantly. Hence, the Bezier extraction is applied to achieve the same shape functions on each element [4]. Within this procedure, supporting points are introduced that change the shape of the ansatzfunctions in the parameter space, but the geometry in global coordinates is kept. The surfaces are defined as a multiplication of two one dimensional functions of the form

$$R_i^p(\xi) = \frac{B_i^p(\xi) w_i}{\sum_{j=1}^{n} B_j^p(\xi) w_j} \ . \tag{6}$$

The Bezier extraction allows the direct evaluation of the B-Splines as Bernstein polynomials

$$B_i^p(\xi) = \frac{1}{(b-a)^p} \binom{p}{i} (\xi - a)^i (b-\xi)^{p-i} \ . \tag{7}$$

The positioning of the collocation points has a large influence on the solution accuracy. For the presented ansatzfunctions, the placement is independent of the shape functions, since the functions are not interpolatory at the collocation points. This is in contrast to a Lagrange formulation, where the ansatzfunction is 1 at the corresponding supporting point. In the presented formulation with discontinuous elements, the zeros of the Legendre polynomials are chosen as supporting points. An optimal placement of these points is a topic for future research.

Numerical Integration

The IGA concept allows an exact representation of the geometry and therefore can increase the solution accuracy significantly. In previous formulations, it was satisfying to compute all the values in the same range as the discretization error of the geometry. Due to the elimination of this approximation, the other influences on the solution quality play a more considerably role. One of the influences is the numerical integration, which has an extraordinary impact on the solution quality in the BEM. The CBIE and HBIE include weakly singular, strongly singular and hypersingular terms that have to be treated carefully. If the collocation point and the source element are well separated, a conventional gaussian integration is applicable. When they are on the same element, the integration becomes singular. A further important type is the quasi singular integration, where the collocation is very close to the source element, but is not lying on it. A difficulty of this type is that no shared local coordinate system exist. For the IGA, the curved surfaces increase this difficulty once more. The integration routines are based on [3] and the main changes are adressed in the following.

The singular integration is done in the sense of Guiggiani's formulation [6]. It removes the singularity by a subtraction of a series expansion from the kernel. This is possible, since the series expansion has the same order of singularity as the kernel and approximates the static fundamental solution. Afterwards, the subtracted term is added back and can be integrated semi-analtically. The procedure leads to a direct evaluation of all singularities for two collocation points lying on the same element. The changes of the ansatzfunctions to NURBS have to be adressed in the required derivatives of the series expansion.

The treatment of the quasi-singular integrals is a sinh-transformation [7], that reduces the singular character. The different local coordinates systems of the two considered collocation points are the challenge of this procedure. The point on the source element with the shortest between the collocation point and source element is needed in local coordinates. For plane Lagrange elements the transformation onto the other local coordinate system, firstly, an orthogonal projection is used and, secondly, the non-linear transformation from local to global coordinates has to be inversed. The inversion is solved by a multi-dimensional Newton-Raphson method. Due to the curved element description by NURBS, the orthogonal projection can not be applied. Therefore, the approximation of the Newton-Raphson method is already used to achieve the point of the shortest distance in global coordinates.

Numerical Examples

In this section the behavior of the IGA in terms of the numerical integration is shown. The first example shall show the influence of the integration onto the CBIE and the second example focuses on the HBIE. In the first example the sinh-transformation is not applied. The results are rated in terms of the Dirichlet-error

$$e_D = \frac{\| (\mathbf{p}_{num} - \mathbf{p}_{ref}) \|}{\|\mathbf{p}_{ref}\|} \tag{8}$$

with

$$\|\mathbf{p}\| = \sqrt{\int_\Gamma |\mathbf{p}|^2 \mathrm{d}\Gamma}\ . \tag{9}$$

The error is integrated over the elements to take the higher ansatzfunctions into account. This error is in general more strictly than a pointwise error. The reference solution depends on the example.

In figure 1 and figure 2 the distributions of the error over the surfaces are plotted. For the two NURBS results the integration order is increased from 4×4 to 20×20. It is visible that an increase leads to a significant improvement of the solution quality and an accuracy near the machine precision is possible. The results for the Lagrange formulation are computed with the higher integration order of 20×20. The results show a fast convergence of the error to a value in the range of the geometrical imperfections. As a consequence, it can be stated that the numerical integration plays a less important rule for Lagrange formulations. It is sufficient to have an accuracy in the same order as the geometrical imperfection. Hence, the Lagrange formulation has a limitation in contrast to the new IGA formulation.

The second example is a cat's eye with an excitation based on a monopole source at $\mathbf{x_{mono}} = [-0,3 \,/\, -0,3 \,/\, -0,3]$, which allows the computation of an analytical solution $\mathbf{p}_{ref}$. The model contains 10 NURBS elements and the computation relies on a 3 times h-refined description. Due to the IGA concept, the elements have the same ansatzfunctions for the geometry and the field variables, which are quadratic in the given case. In figure 3 the error over the frequency is plotted for the CBIE, the BM and the BM with the sinh-transformation. The BM shows the correct regularization of the spurious eigenfrequencies, but with the drawback of an overall increase of the error.

The additional use of the sinh-transformation reduces the error significantly. This shows the necessity of special integration routines for the HBIE. Beside the singular integrations, the quasi singular integrations become more important if more collocation points are located on one element, which is the case for an IGA with higher order ansatzfunctions for the field variables. In figure 4 the error distribution with and without the sinh-transformation is shown for $f = 1,000$ Hz, where high errors are located at collocation points near an edge.

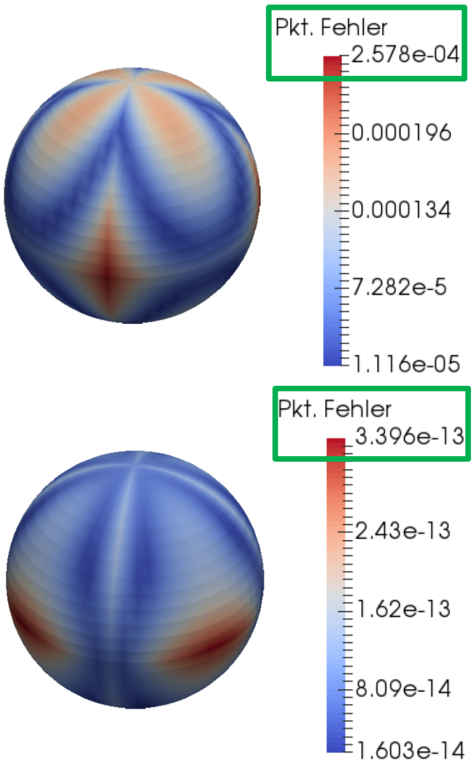

Figure 1: Error distribution over the sphere with a NURBS formulation with an increased integration order from 4×4 to 20×20

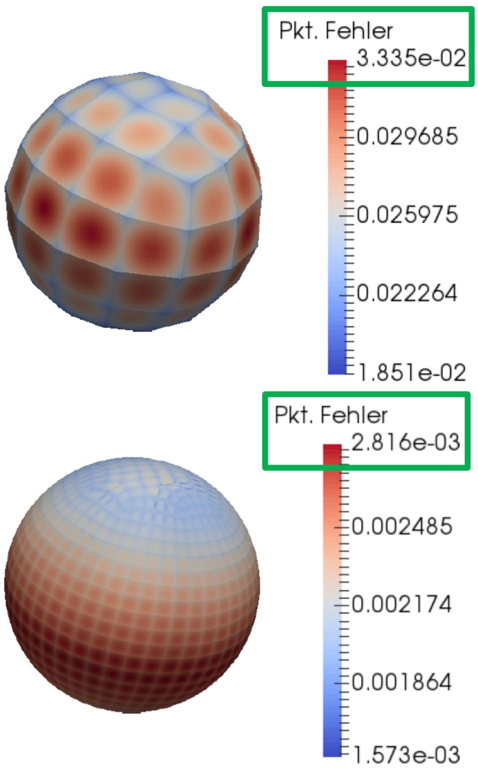

Figure 2: Error distribution over the sphere with a Lagrange formulation and an increase of the elements

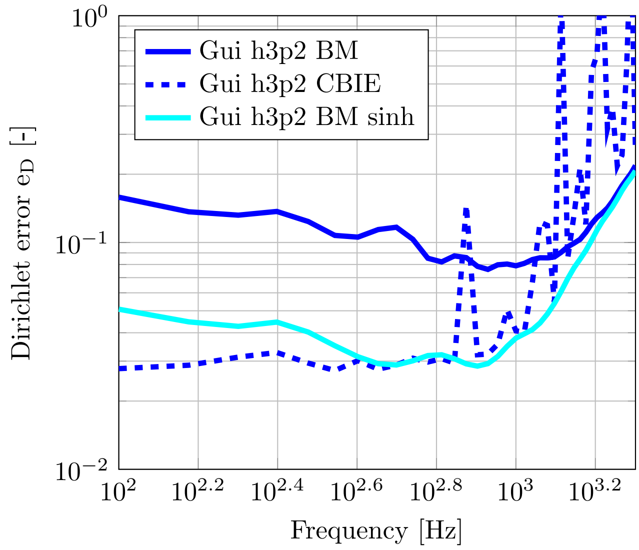

Figure 3: Error over frequency

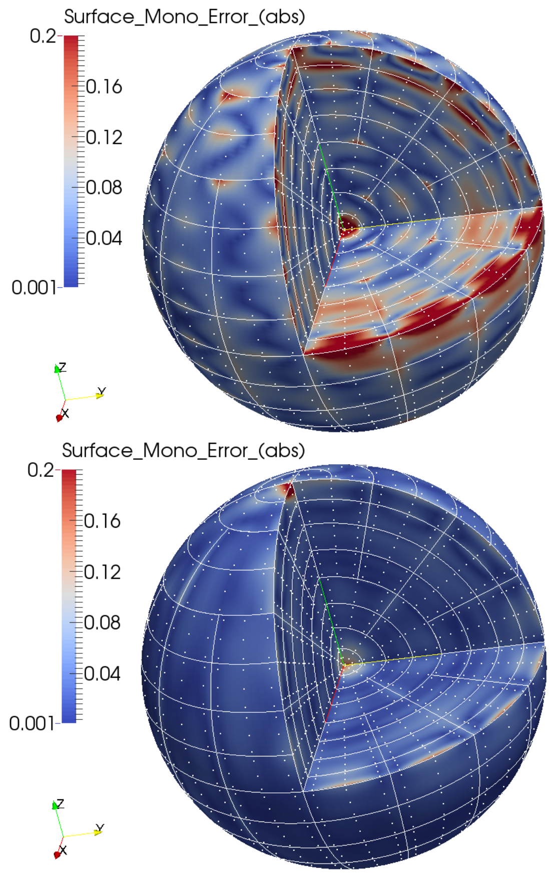

Figure 4: Error distribution over the cat's eye without and with sinh-transformation

Conclusion

In the presented contribution the importance of the numerical integration for the Isogeometric Boundary Element Method is shown. The basic equations of the BEM, the IGA and the required integration routines are explained. Special treatment is necessary for the incorporation to the new IGA concept. The importance is confirmed by some representative examples. For the conventional boundary integral equation the machine precision of the computer can be achieved with the new formulation. For the hypersingular case, the correct handling of the singular integrals is required to overcome the problem of the spurious eigenfrequencies. Additionally, for more collocation points on one element, which arises in an Isogeometric BEM, the quasi singular integrals become more important. The presented methodology allows an accurate solution of acoustical problems on exact geometries. The application of these procedures to more realistic, engineering problems is one of the most important next steps.

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