Sparse Grid Application in the Stochastic Physics-based Model of the Piano Soundboard

Dóra Jenei-Kulcsár[1], Péter Fiala[2]

[1] *Budapest University of Technology and Economics, H-1111 Budapest, Hungary, E-mail: dkulcsar@hit.bme.hu*

[2] *Budapest University of Technology and Economics, H-1111 Budapest, Hungary, E-mail: fiala@hit.bme.hu*

Introduction

Piano soundboards are one of the most important part of the sound production. The geometry and material effect the sound quality by inter-string coupling and sound radiation. Besides the traditional wood resonators modern composite laminas also have stochastic parameters through variation of fibre orientation and thickness. That is why modelling the uncertainty plays a key role in the understanding of the produced sound.

In the literature there are numerous article how to model a piano soundboard using different approaches (e.g. [3], [10]), but in all cases the material properties are handled as deterministic. To model the stochastic behaviour there exist some models for composite structures (e.g. [9]), in which cases the computation effort can be drastically reduced applying sparse grids (e.g. [11]).

In this paper we present a stochastic finite element model of the laminated piano soundboard that is capable to assess the uncertainty of its lower eigenfrequencies and mode shapes resulting from uncertain fibre directions. The stochastic soundboard model is integrated in a piano simulation tool, so the effects of chosen models and parameters are examinable in detailed simulation.

In the next sections we briefly introduce all used model and present numerical simulation results.

Piano Model Overview

Since the piano have hundreds of mechanical element on the path from touch to sound radiation, we use a reduced model of the instrument containing hammers, strings and the soundboard.

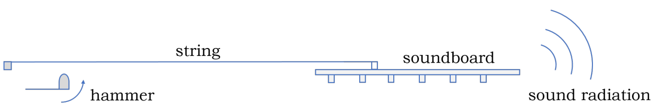

Figure 1: Abstract piano model.

Hammers are modelled as hysteretic mass-spring systems with multiple springs coupled to the mass (e.g. [2]). In our **string** model a multi-string digital waveguide is applied. The effect of the soundboard as well as the losses and damping caused by material properties and air contact are handled using IIR filter sets (s.a. [1]). The system is terminated by a near-field **sound radiation** model based on the Rayleigh-integral using a FIR filter set. (For a more detailed model description s.a. [7].)

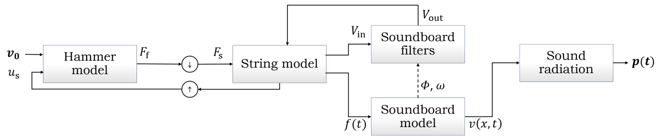

Figure 2: Piano model overview – block diagram.

Soundboard Model Overview

Piano soundboards are complex structures in which the main resonator (plate) are stiffened and slightly bowed with ribs on the bottom, while on the upper side the bridges are installed. The main resonator traditionally is built from solid music wood, but nowadays laminated woods and composites are also in use. To be able to incorporate the effect of such a structure in the presented piano model, the modelling goal is to determine the modal description of this structure. The elastic material model is handled separately from the finite element model of the geometry for solid elements and lamina with and without ribs.

The **material model** is based on the generalised Hooke's law valid for orthotropic materials. The resulting equation of forces and moments is derived from the Kirchoff-Love thin plate theory. By determining the material matrix a transformation from material coordinate system (μ_i) to the coordinate system of the geometry (x_i) has to be performed. The numerical model of the geometry is

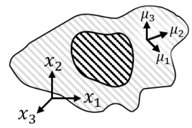

Figure 3: Material model and the geometry.

based on the **finite element approach** according to the dynamics of an elastic volume (Ω) given by arbitrary boundary (Γ) is determined through discretisation of the geometry and function space. The volume is split into elements ($\Omega_\mathrm{e}, \Gamma_\mathrm{e}$) which are projected to predefined elementary elements ($\Omega_\mathrm{e}^\mathrm{s}, \Gamma_\mathrm{e}^\mathrm{s}$). The solution for displacement (U) equation is given by a superposition of finite number of polynomial base function defined over these standard elements. Outputs of the finite element model are the so-called mass- ($\mathcal{M}$) and stiffness matrices ($\mathcal{K}$). After several steps the **modal description** (eigenfrequencies (ω_k) and mode shapes (Φ_k)) can be determined

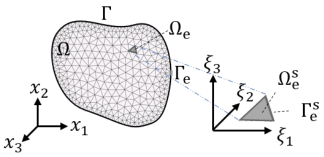

Figure 4: Finite element model.

by solving a standard eigenvalue problem:

$$\left(\mathcal{M}^{-1} \cdot \mathcal{K} - \omega^2 \cdot I\right) \cdot U(\omega, x) = 0\,, \tag{1}$$

where I is the identity matrix. The resulting mode shapes and modal weights (α_{k}) form the required base.

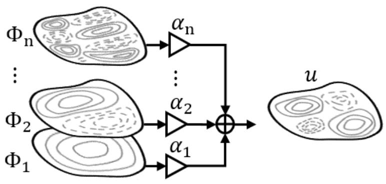

Figure 5: Modal superposition.

In case of **laminated resonators** ply-wise determination of the material matrix is possible. The resulting formula for forces and moments is the same as for a general material, but also the integration has to be performed ply-wise. The steps in finite element model are the same. To reduce the number of used nodes in the

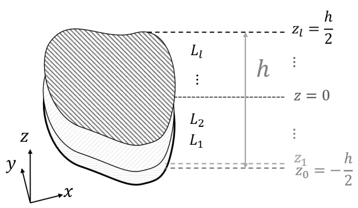

Figure 6: Laminated soundboard model.

numerical model for **ribbed soundboard**, we apply the master-slave multi-freedom constrain theory. We handle the plate as master, the ribs and bridges as slaves. The nodes of slave models (s_{i}) are interpolated ($\mathrm{g}(s_{\mathrm{i}})$) to the master nodes ($m_{\mathrm{i}}$) after independent determination of mass- and stiffness matrices.
For a more detailed soundboard model description s.a. [5].

Stochastic Approach

The uncertain parameters (e.g. fibre orientation) are modelled as **stochastic processes** using the Karhuen-Loève expansion:

$$r(x, \theta) = \sum_{k=1}^{\infty} \xi_{\mathrm{k}}(\theta) \cdot \mathrm{g}_{\mathrm{k}}(x)\,, \tag{2}$$

where $\xi_{\mathrm{k}}(\theta)$ are the coefficients and $\mathrm{g}_{\mathrm{k}}(x)$ are an orthonormal base over space x. In case of a centred stochastic process (expected value of every elements

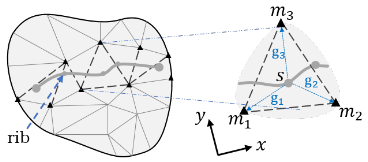

Figure 7: Ribbed soundboard – master-slave model.

equals zero) coefficients are pairwise uncorrelated and the orthonormal base is given by the covariance function ($\mathcal{C}$) of the process:

$$\mathcal{C}(x_1, x_2) = \sum_{k=0}^{\infty} \lambda_{\mathrm{k}} \cdot \mathrm{l}_{\mathrm{k}}(x_1) \cdot \mathrm{l}_{\mathrm{k}}(x_2)\,, \tag{3}$$

where λ_{k} are the squared eigenvalues and $\mathrm{l}_{\mathrm{k}}(x)$ are the normalised eigenvectors of the covariance kernel [4]. Using the eigenvector decomposition the stochastic process has the form:

$$r(x, \theta) = \sum_{k=1}^{\infty} \xi_{\mathrm{k}}(\theta) \cdot \sqrt{\lambda_{\mathrm{k}}} \cdot \mathrm{l}_{\mathrm{k}}(x)\,. \tag{4}$$

In practical cases an arbitrary stochastic process can be rewritten in the form of the centred case and the number of used terms are reduced to the m largest eigenvalues of the covariance kernel. It can be proven that this choice gives the best approximation by definition of the total mean square error.

To apply the stochastic process in the introduced soundboard model, the **stochastic modal description** has to be determined using the stochastic form of the standard eigenvalue problem in (1). It means that every term of the equation depends on the $\xi_{\mathrm{k}}(\theta)$ random variables, because of the stochastic mass- and stiffness matrices.

For a given realisation ($\xi_{\mathrm{k}}(\theta)$) of the matrices the eigenvalue problem can be solved deterministically. Stochastic eigenfrequencies and mode shapes (e.g. expected values, variance) can be determined statistically from prescribed number of realisations.

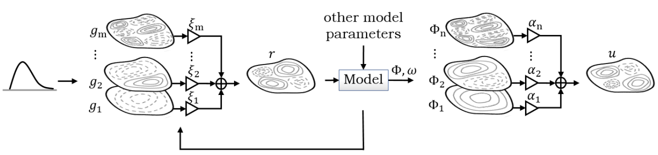

Figure 8: Stochastic soundboard model.

Application of Sparse Grids

Since the modal values are computed using multidimensional numerical integration, the number of required realisations (so-called collocation points) depends on the chosen **solution strategy**. There basically exist the following options: running Monte Carlo simulation, perform systematic sampling or implement some type of quadrature method. In case of **Monte Carlo simu-**

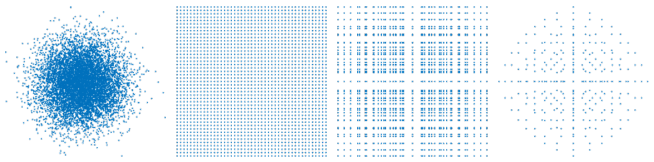

Figure 9: Collocation points in case of Gaussian stochastic parameter: Monte Carlo (l.), systematic sampling (m.l.), full (m.r.) and sparse grid using Smolyak's quadrature (r.) based on Gauss-Hermite quadrature.

lation random variables with given distribution are selected. The benefit of this choice is the easy statistical interpretation, but it converges slowly to the solution. Performing **systematic sampling** means a modest reduction of collocation points, while the previous knowledge of the parameters are lost. In lower dimensional cases (few number of stochastic parameters) can be efficient. For higher order problems quadrature strategies are widely used. In case of **full grids** one dimensional quadratures are independently determined by the distribution of stochastic parameters. The collocation points are nodes of the full grid spanned by these quadratures. Although the parameters are optimally modelled, the number of selected points drastically increases by added dimensions (e.g. number of stochastic parameters or layers). **Sparse grid methods** try to avoid this problem by carefully addition of collocation points.

One of the most effective sparse grid configuration is given by the **Smolyak's quadrature**. Based on the one dimensional quadratures ($U_\mathrm{k}^{(\mathrm{j})}$) difference quadratures are defined by:

$$\Delta_0^{(\mathrm{j})} = 0 \quad \Delta_1^{(\mathrm{j})} = U_1 \quad \Delta_{\mathrm{k}+1}^{(\mathrm{j})} = U_{\mathrm{k}+1}^{(\mathrm{j})} - U_\mathrm{k}^{(\mathrm{j})}\,, \tag{5}$$

and the d-dimensional quadrature of order n is computed by the formula:

$$Q_\mathrm{n}^\mathrm{d} = \sum_{\substack{\|\alpha\|_1 \leq n \\ \alpha \in \mathbb{N}^d}} \bigotimes_{j=0}^{d} \Delta_{\alpha_\mathrm{j}}^{(\mathrm{j})}\,, \tag{6}$$

where

$$\|\alpha\|_1 = \sum_{j=1}^{d} |\alpha_\mathrm{j}|\,. \tag{7}$$

The number of resulting collocation points are optimal but the quadrature are symmetric for all stochastic parameters (s.a. [6]).

Following the widening sampling range by increasing dimension and quadrature order, mode order exchanges in simulation results, causing error by the evaluation of statistical computations. It can be solved by using proper **mode tracking** techniques.

Numerical Example

The presented numerical example is a patch-like spruce plate. Material properties are based on [8]. The plate

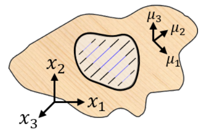

Figure 10: Example geometry and material.

has the dimensions 1.56×1.45 m and thickness of 7 mm. The orientation of the fibres (μ_1) are -45°. The first three

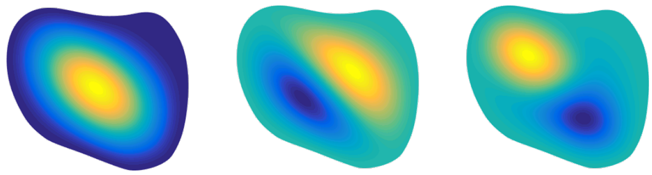

Figure 11: Solid plate modes at 19, 31 and 45 Hz.

dynamic modes of our example is presented in Figure 11.

At first step the geometry is split into seven layers of equal thickness. The fibre orientations are randomly selected. The eigenfrequencies are shifted into 20, 35 and 48 Hz. The resulting mode shapes are presented in Figure 12. Although the fibre orientation varies in a wide range

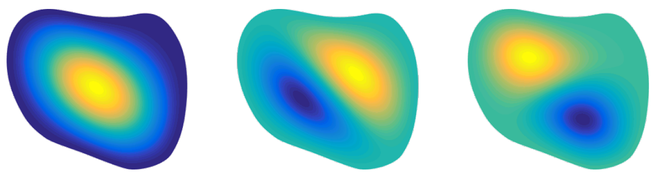

Figure 12: Laminated plate modes at 20, 35 and 48 Hz.

([-75°; -15°]) within the layers, the end effect is modest. The layered structure compensate the differences.

As second steps beech ribs are added to the layered geometry. Every rib is 1 cm wide, on the middle has thickness of 3 mm and near the boundary 1 mm. The orientation is quite normal to the fibre orientation (μ_2). The resulting eigenfrequencies are 28, 54 and 66 Hz. The ribs are dominant in the radial direction wich can be well seen in the mode shapes on Figure 13.

In the case of a solid, not ribbed plate with stochastic fibre orientation with expected value of $-45°$ and standard deviation of 5° using the seven largest modes from the Karhuen-Loève expansion the eigenfrequencies are shifted upwards and the expected value of mode shapes are a bit changed compared to the case of deterministic simulation with standard deviation about 9°. The end effect is shown in Figure 14. Installing ribs in this case shows again that the ribs became dominant over the effect of stochastic variation. The mode shapes are similar as in laminated case (s.a. Figure 15). The standard deviation is decrease a bit, but the difference is negligible.

Conclusion and Future Work

We presented a stochastic soundboard model applying sparse grids for reduction of required collocation points.

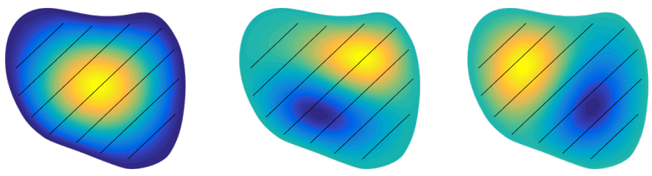

Figure 13: Laminated and ribbed plate modes at 28, 54 and 66 Hz.

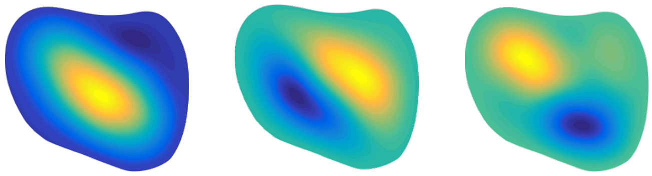

Figure 14: Solid plate modes at 32, 40 and 65 Hz with stochastic fibre orientation (standard deviation: 5°).

The soundboard is integrated in a MATLAB-based simulation framework, so the effect of modifications are direct examinable in the radiated sound.

Although stochastic material parameters exist naturally and as a result of manufacturing uncertainties, the most effect can be compensated by increasing the number of layers or using the ribbed structures.

In the radial direction of the material the stiffening effect of the ribs becomes dominant, so quite every other effect material variability disappears.

In the future we plan to examine the effect of other stochastic parameters like layer thickness or material density and to run higher order simulation for getting more precise results.

References

[1] Bank, B. *Physics-based Sound Synthesis of String Instruments Including Geometric Nonlinearities.* PhD thesis, Budapest University of Technology and Economics Department of Measurement and Information Systems, 2006.

[2] Bensa, J., Gipouloux, O., and Kronland-Martinet, R. Parameter fitting for piano sound synthesis by physical modeling. *The Journal of the Acoustical Society of America 118*, 1 (2005), 495.

[3] Chabassier, J. *Modélisation et simulation numérique d'un piano par modèles physiques.* PhD thesis, Ecole polytechnique de Paris, 2012. Thèse de doctorat dirigée par Joly, Patrick Mathématiques appliquées Palaiseau, Ecole polytechnique 2012.

[4] Ghanem, R. G., and Spanos, P. *Stochastic Finite Elements: A Spectral Approach.* Springer, 2011.

[5] Jenei-Kulcsár, D., and Fiala, P. Assessing the effect of laminate soundboard characteristics in the physics-based model of the piano. In *Tagungsband DAGA 2017* (2017).

[6] Kaarnioja, V. Smolyak quadrature. Master's thesis, University of Helsinki Department of Mathematics and Statistics, May 2013.

[7] Kulcsár, D., and Fiala, P. A compact physics-based model of the piano. In *Proceedingd of DAGA 2016* (2016).

[8] Ross, R. J., and Service., F. P. L. U. F. Wood handbook : wood as an engineering material. Tech. rep., 2010.

[9] Sepahvand, K. Spectral stochastic finite element vibration analysis of fiber-reinforced composites with random fiber orientation. *Composite Structures 145* (jun 2016), 119–128.

[10] Trévisan, B., Ege, K., and Laulagnet, B. A modal approach to piano soundboard vibroacoustic behavior. *The Journal of the Acoustical Society of America 141*, 2 (feb 2017), 690–709.

[11] Yadav, V. *Novel Computational Methods for Solving High-Dimensional Random Eigenvalue Problems.* PhD thesis, University of Iowa, 2013.

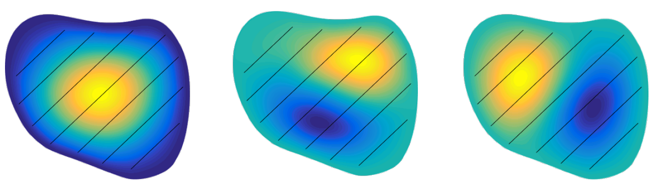

Figure 15: Solid ribbed plate modes at 26, 50 and 63 Hz with stochastic fibre orientation (standard deviation: 5°).