Bayesian Inference Method to Identify Random Parameters

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Introduction

Response quantification for aerospace and automobile systems is not always deterministic due to inherent randomness in system parameters. Nowadays, fiber reinforced composite (FRC) has become an ideal material to the aerospace and automobile industries owing to its high strength to weight ratio. However, such material shows a wide range of the variability in structural response due to the inherent randomness in elastic moduli, damping, fiber orientation, etc. Therefore, uncertainty quantification of the FRC has been a major challenge in the aerospace and automobile industries in achieving reliable structural design.

In the recent years, spectrum-based general polynomial chaos (gPC) expansion method has received significant attention in the industries over the classical sampling-based Monte Carlo (MC) simulation method due to faster rate of convergence and computational efficiency [1]. The application of the gPC expansion method to analyze the FRC plate under static and dynamic excitations has already been reported [2, 3]. In these works, randomness due to the elastic moduli has been considered. A good agreement of the response has been reported in the distributions obtained using the gPC expansion method as compared to the MC simulation method [1]. A successful representation of the uncertainty in the system response has been reported using the gPC expansion method, which depends on identification of probability density function (PDF) for the input parameters in the stochastic finite element method (SFEM). Rosić et al. [4] presented a non-sampling-based Bayesian updation technique on prior information of the random parameters in combination with the gPC expansion. They applied this method in Darcy-like flow through porous media. Mehrez et al. [5] have reported the characterization of stochastic properties of the elastic moduli of the heterogeneous composite fabrics from limited experimental data using gPC expansion method. They considered coefficients of the gPC expansion as random and their distributions have been obtained using the Bayesian inference method from the limited experimental data. Application of the Bayesian inference method for the uncertain input parameters of the FRC plate using the gPC expansion method requires further investigation.

The purpose of the present work is to infer the posterior density function of the random input parameters i.e. elastic moduli of the FRC plate, through the Bayesian inference method and apply the posterior distribution to the gPC expansion method to study the uncertainty in the frequency responses of the FRC plate under free vibration. The identification of the random space of the uncertain input parameters is a crucial aspect before application of the gPC expansion method. The present work shows the application of successful identification of the posterior density function of the elastic moduli using Pearson model. The major objectives of the present study are: (1) to identify random space of uncertain input parameters; (2) to identify the posterior density function of frequency responses of the FRC plate; and (3) to investigate the effect of application of the Bayesian inference on the distribution of the frequency responses of the FRC plate.

Bayesian Inference

The Bayesian inference method is used here to postulate the data for elastic moduli of the FRC from prior uncertain knowledge. It is assumed that the elastic moduli follow a log-normal ($\mathcal{LN}$) distribution [2]. Independence Jeffreys' prior is used for the Bayesian inference, considering the mean (μ) and standard deviation (σ) as unknowns [6]. The independence Jeffreys' prior follows the Fisher information matrix, where μ and $\log\sigma^2$ are independent and uniformly distributed. By combining the likelihood function and the prior density function, the joint posterior density function is represented as

$$P(\theta|\mu,\sigma^2) = (2\pi\sigma^2/n)^{-1/2}\exp[-(n/2\sigma^2)(\mu-\hat{\mu})^2] \times (\gamma\hat{\sigma}^2/2)\left[\frac{(\sigma^2)^{-(\gamma+2)/2}\exp(-\gamma\hat{\sigma}^2/2\sigma^2)}{\Gamma(\gamma/2)}\right] \quad (1)$$

where $\hat{\mu} = (1/n)\sum_1^n y_i$, $\gamma = n-1$, $y_i = \ln(x_i)$, and $\hat{\sigma}^2 = [1/(n-1)]\sum_1^n (y_i - \hat{\mu}^2)$.

Stochastic Finite Element Method for FRC

Orthotropic FRC plate is mathematically modelled by finite element (FE) method considering first order shear deformation theory (FSDT). Commercial FE package Ansys is used to model the FRC plate using 8-node plate element SHELL281 having six degrees of freedom per node. This FE model is used to deterministically solve the problem in the subsequent SFEM steps. The elastic moduli of the FRC plate are considered as uncertain parameters and represented by a truncated gPC expansion as [1],

$$E(\xi) = \sum_{i=0}^{N} E_i \psi_{ie}(\xi_e) \quad (2)$$

where E_i are the deterministic coefficients, and $\psi_{ie}(\xi_e)$ is orthogonal basis function in the gPC expansion method. One dimensional 3rd order gPC expansion is used to generate the random elastic parameters, i.e. E_{11}, E_{12}, G_{12}, and G_{23}. The selection of orthogonal polynomial basis function depends upon the probability space in which the random parameters belong. The prior distribution of the elastic moduli is assumed to be distributed log-normally. For log-normal random parameters, Hermit polynomial is used as an orthogonal basis function. The random frequency responses of the FRC plate are also approximated by a truncated multidimensional gPC expansion as,

$$F(\xi) = \sum_{i=0}^{N} F_i \psi_{if}(\xi_f) \tag{3}$$

where $\psi_{if}(\xi_f)$ is the multidimensional polynomial basis function, and F_i are the deterministic unknown coefficients. The deterministic unknown coefficients of response are calculated by stochastic collocation methods [1]. For 3rd order gPC expansion with two-dimensional random vectors i.e. $\xi_f = \left(\xi_{f_1}, \xi_{f_2}\right)$ requires determining 10 numbers of deterministic coefficients. A set of deterministic solution of the system is generated from the N sets of the random elastic parameters. Eq. 3 is then solved to determine the deterministic coefficients for the N numbers of collocation points.

Identification of the Uncertain Parameters

The PDF identification of the elastic parameters is a key issue for successful representation of synthetic PDF of system response employing the gPC expansion method. The most common and widely accepted method for PDF identification is the Pearson model [7]. The method involves calculation of terms β_1 (squared skewness) and β_2 (traditional kurtosis) from the digitally generated samples of uncertain parameters and plot on the Pearson diagram, where

$$\beta_1 = \mu_3^2 / \mu_2^3, \beta_2 = \mu_4 / \mu_2^2 \tag{4}$$

and μ_k is the k^{th} central moment about the parameter mean of the uncertain parameter is given by,

$$\mu_k = (1/N\sigma^k) \sum_{i=1}^{N} [p_i - E(P)]^k \tag{5}$$

where σ is the standard deviation of the generated sample space, p_i are the generated samples, and $E(P)$ is the expected value of the samples. Pearson family [7] contains many popular standard PDFs i.e. Normal, Log-normal, Beta, Gamma, etc. as well as some non-standard type PDFs also. The Pearson model is able to identify PDF of random parameters with high accuracy from the precisely calculated terms β_1 and β_2.

Numerical Case Study

A study on glass fiber-epoxy, FRC plate with free-free boundary condition is presented here to ascertain the applicability of the Bayesian inference in investigating the uncertainty of the system response. A 12-layer symmetric cross-ply laminate glass fiber epoxy composite is used for the investigation. Dimensions of the plate are 250 mm × 125 mm × 2 mm. As stated previously, prior uncertain elastic parameters are distributed log-normally; and mean value and standard deviation of the parameters are as given in Table 1. The 3rd order gPC expansion method with Hermit polynomial function is employed to represent the prior density function of the input parameters. The reconstructed prior is shown in Fig. 1 in comparison with the theoretical log-normal PDF. It is found that the 3rd order gPC expansion is able to represent the distribution with high accuracy. The parameters of the posterior density function of the elastic moduli are estimated using the Bayesian inference with reference to Eq. 1 and posterior density function has been represented by the gPC expansion method (Fig. 1).

Table 1: Material parameters

Parameters	Values
Elastic moduli [GPa]	$E_{11} = \mathcal{LN} \sim (47.451, 12)$ $E_{22} = \mathcal{LN} \sim (9.739, 3)$ $G_{12} = \mathcal{LN} \sim (4.047, 1)$ $G_{23} = \mathcal{LN} \sim (2.0235, 0.7)$
Poisson's ratio	0.24
Density [kg/m^3]	2174.24

The class of posterior PDF is needed to be identified before representation of the posterior distribution of the system response by the gPC expansion method. The Pearson model is used here to identify the sample space of the posterior data of the elastic moduli. Thus, corresponding values of the Pearson parameters β_1 and β_2 are calculated from the posterior PDF according to Eq. 4 and Eq. 5. The posterior distribution parameters and Pearson parameters of elastic moduli are presented in Table 2. The Pearson parameters of the elastic moduli are represented in Fig. 2 in terms of β_1 and β_2. It is found that the uncertain parameters are near to the coordinate $(\beta_1, \beta_2) = (0,3)$ of the Gaussian distribution. In Fig. 2, line VII represents the log-normal type of distribution and the coordinate points (β_1, β_2) of the elastic moduli are in the close vicinity of line VII [7]. In such scenario, all the posterior PDFs of the uncertain elastic parameters are treated as log-normally distributed. Thereby, Hermit polynomial is used as an orthogonal basis function for the third order gPC expansion to represent the posterior density function of the system response. As observed from Fig. 1, the synthetic posterior PDFs of all the elastic moduli are comparable with the theoretical log-normal PDF.

To construct distribution function of eigen frequency of the FRC plate under free vibration, two-dimensional 3rd order gPC expansion is employed here. Roots of the one higher order orthogonal Hermit polynomial, i.e. ±2.334 and ±0.7420 are used as collocation points along with the typical collocation point 0. A set of $5^2 = 25$ collocation points is generated to estimate the unknown coefficients (Eq. 3) of the first four eigen frequency distributions. The typical collocation points in an increasing order are: (0,0), (0,0.7420), (0.7420,0), etc. After solving the values of unknown deterministic coefficients a set of 1000 random first four eigen frequencies are generated by considering values of polynomial, $\psi_{1f}(\xi_f)$, $\psi_{2f}(\xi_f)$, …, $\psi_{10f}(\xi_f)$, where $\xi_f = (\xi_{f1}, \xi_{f2}) = \mathcal{N}(0,1)$ are normally distributed.

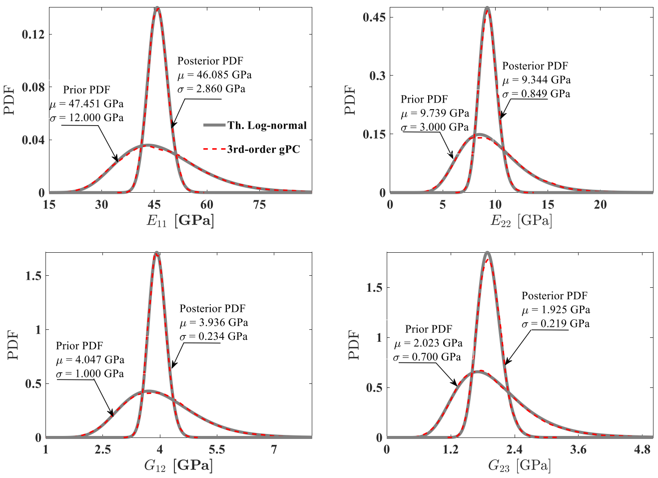

Figure 1: Synthetic prior and posterior PDF of elastic moduli E_{11}, E_{22}, G_{12}, and G_{23} [GPa] from 3rd order gPC expansion (dashed line) compared to theoretical log-normal PDF (bold line)

Table 2: Parameters of posterior PDF of elastic moduli

Parameters	μ [GPa]	σ [GPa]	β_1	β_2
E_{11}	46.085	2.860	0.035	3.035
E_{22}	9.344	0.849	0.076	3.119
G_{12}	3.936	0.234	0.046	3.114
G_{23}	1.925	0.219	0.103	3.203

The first four deterministic coefficients of the prior and posterior gPC expansion methods for first four eigen frequencies are presented in Table 3. Mean and standard deviation of the prior and posterior distributions are presented in Table 4. It is evident from Table 4 that dispersion of the distribution of the eigen frequencies is increasing with the increase in eigen frequency mode for both prior and posterior distributions. However, the range of dispersion in the posterior eigen frequency distribution is reduced considerably in comparison with the prior distributions. Introduction of the Bayesian inference method within the gPC expansion method greatly reduces the range of dispersion of the distribution for the posterior frequency responses of the FRC plate. Considering mean value of the elastic moduli as deterministic, the first four eigen frequencies of the FRC plate with free-free boundary conditions are: 91.287, 108.98, 215.54, and 300.32 Hz. It is evident from Table 4 that the posterior mean values of the first four eigen frequencies are close to the deterministic values in comparison with the mean values of the prior eigen frequency distributions. It is observed that the gPC coefficients (Table 3) converge rapidly for the posterior eigen frequencies over the prior eigen frequencies. The rapid convergence of the gPC coefficients is attributed to the lower range of variation in the standard deviation (Table 4) for the posterior eigen frequency distribution. Nevertheless, the values of the standard deviations are in increasing order with the frequency mode for both prior and posterior eigen frequency distributions. The PDFs of the prior and posterior eigen frequencies are presented in Fig. 3 and Fig. 4 in comparison with the kernel density PDF. It represents accuracy of the gPC expansion method as compared to the simulated eigen frequency distribution with reference to the kernel density PDF. Identification for class of the posterior frequency distributions is carried out from the Pearson model. Pearson parameters, β_1 and β_2 are calculated from the digitally simulated data using the gPC expansion and presented in Table 4.

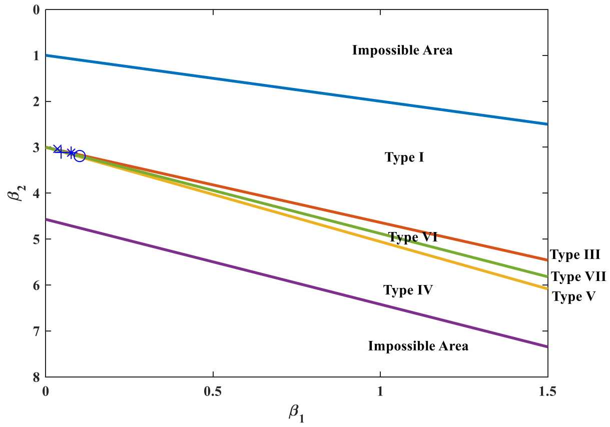

Figure 2: The Pearson diagram comparing the type of posterior PDF for uncertain elastic moduli (× - E_{11}, * - E_{22}, + - G_{12}, and o - G_{23})

It is seen from the Pearson diagram shown in Fig. 5 that the Pearson coordinate points β_1 and β_2 for the posterior frequency responses are quite close to the coordinate, $(\beta_1, \beta_1) = (0,3)$ of the normal distribution. Thus, the posterior distribution of the frequency responses are identified with the generalized normal distribution with prescribed mean and standard deviation.

Table 3: The coefficients of gPC of prior and posterior distribution for first 4-eigen frequencies for FRC plate

	Mode	gPC coefficients			
		F_0	F_1	F_2	F_3
Prior	1	90.209	11.486	0.000	0.632
	2	107.445	14.763	0.000	0.868
	3	212.865	27.662	0.000	1.551
	4	296.100	40.701	0.000	2.392
Posterior	1	89.963	2.703	0.000	0.040
	2	107.395	3.668	0.000	0.066
	3	212.406	6.616	0.000	0.104
	4	295.962	10.145	0.000	0.108

Conclusions

The stochastic-based gPC expansion method in accordance with the Bayesian inference is presented in this work to investigate the posterior distribution of the uncertain elastic parameters and classify distribution of the posterior frequency response of the FRC plate under free vibration. The analysis presents a basic idea in successfully identifying classes of the uncertain parameters, suitably applying the Bayesian inference.

Table 4: Distribution parameters and Pearson coefficients for first four prior and posterior eigen frequencies

	Mode	μ	σ	β_1	β_2
Prior	1	89.935	11.413	0.117	3.197
	2	107.322	14.786	0.171	3.264
	3	231.370	27.832	0.137	3.302
	4	297.447	41.454	0.168	3.222
Posterior	1	90.067	2.754	0.009	3.035
	2	107.580	3.815	0.007	2.978
	3	212.372	6.511	0.012	3.039
	4	296.270	9.705	0.002	2.956

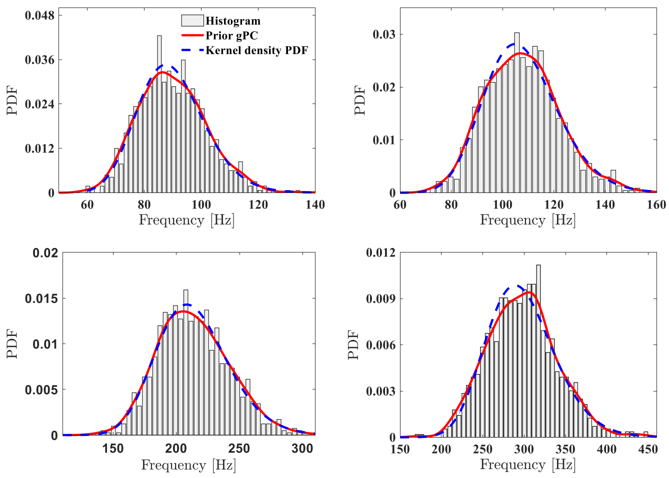

Figure 3: Prior gPC distribution of first four eigen frequencies [Hz] (bold line) compared to kernel density PDF (dashed line)

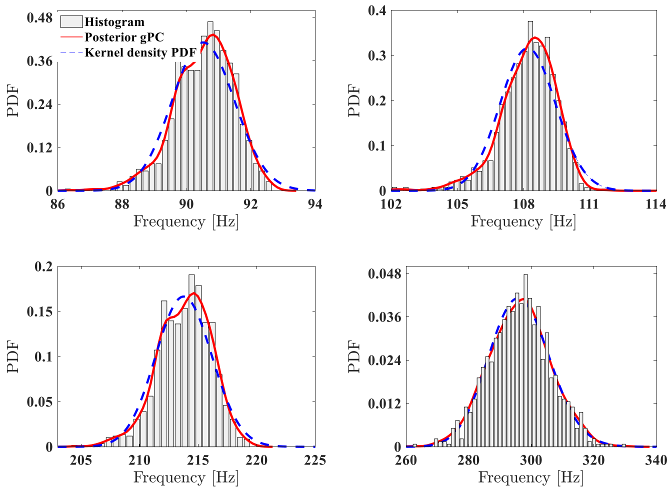

Figure 4: Posterior gPC distribution of first four eigen frequency [Hz] (bold line) compared to kernel density PDF (dashed line)

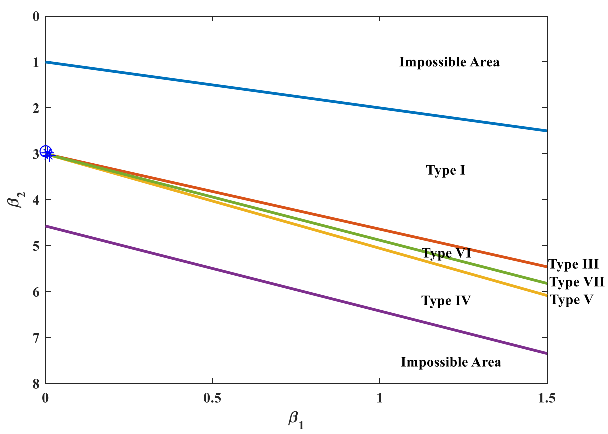

Figure 5: The Pearson diagram comparing the type of posterior distribution for first four eigen frequencies (× - 1st eigen freq., * - 2nd eigen freq., + - 3rd eigen freq. and o - 4th eigen freq.)

It is concluded that dispersion of the posterior distribution of the eigen frequency response is less in comparison with the prior eigen frequency responses. Thus, Bayesian inferred responses present a low level of uncertainty, which lead to reliable design of the FRC plate for various applications.

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